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Operations on Submodules in Left Module over Associative Ring [1]

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Summary. Definition of sum, direct sum and intersection of submodules. We prove a number of theorems related to these notions. This article originated as a generalization of the article [10].

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The terminology and notation used here are introduced in the following papers: [1], [12], [14], [9], [8], [13], [2], [11], [7], [3], [4], [5], and [6]. For simplicity we adopt the following rules: R denotes an associative ring, V denotes a left module over R, W, W_1, W_2, W_3 denote submodules of V, u, u_1, u_2, v, v_1, v_2 denote vectors of V, and x is arbitrary. Let us consider R, V, W_1, W_2. The functor $W_1 + W_2$ yields a submodule of V and is defined by:

(Def.1) the carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider R, V, W_1, W_2. The functor $W_1 \cap W_2$ yielding a submodule of V is defined by:

(Def.2) the carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

One can prove the following propositions:

(1) The carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

(2) If the carrier of the carrier of $W = \{v + u : v \in W_1 \wedge u \in W_2\}$, then $W = W_1 + W_2$.

(3) The carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

(4) If the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2), then $W = W_1 \cap W_2$.

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(5) $x \in W_1 + W_2$ if and only if there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(8) $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = W_1 + W_2 + W_3$.

(11) W_1 is a submodule of $W_1 + W_2$ and W_2 is a submodule of $W_1 + W_2$.

(12) W_1 is a submodule of W_2 if and only if $W_1 + W_2 = W_2$.

(13) $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

(15) $\Omega_V + W = V$ and $W + \Omega_V = V$.

(16) $\Omega_V + \Omega_V = V$.

(17) $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = W_1 \cap W_2 \cap W_3$.

(20) $W_1 \cap W_2$ is a submodule of W_1 and $W_1 \cap W_2$ is a submodule of W_2.

(21) W_1 is a submodule of W_2 if and only if $W_1 \cap W_2 = W_1$.

(22) If W_1 is a submodule of W_2, then $W_1 \cap W_3$ is a submodule of $W_2 \cap W_3$.

(23) If W_1 is a submodule of W_3, then $W_1 \cap W_2$ is a submodule of W_3.

(24) If W_1 is a submodule of W_2 and W_1 is a submodule of W_3, then W_1 is a submodule of $W_2 \cap W_3$.

(25) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(26) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(27) $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(28) $\Omega_V \cap \Omega_V = V$.

(29) $W_1 \cap W_2$ is a submodule of $W_1 + W_2$.

(30) $W_1 \cap W_2 + W_2 = W_2$.

(31) $W_1 \cap (W_1 + W_2) = W_1$.

One can prove the following propositions:

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a submodule of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a submodule of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a submodule of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a submodule of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If W_1 is a submodule of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) $W_1 + W_2 = W_2$ if and only if $W_1 \cap W_2 = W_1$.

(38) If W_1 is a submodule of W_2, then $W_1 + W_3$ is a submodule of $W_2 + W_3$.

(39) If W_1 is a submodule of W_2, then W_1 is a submodule of $W_2 + W_3$.

(40) If W_1 is a submodule of W_3 and W_2 is a submodule of W_3, then W_1+W_2 is a submodule of W_3.

(41) There exists W such that the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cup$ (the carrier of the carrier of W_2) if and only if W_1 is a submodule of W_2 or W_2 is a submodule of W_1.

Let us consider R, V. The functor $\mathrm{Sub}(V)$ yields a non-empty set and is defined by:

(Def.3) for every x holds $x \in \mathrm{Sub}(V)$ if and only if x is a submodule of V.

In the sequel D denotes a non-empty set. One can prove the following three propositions:

(42) If for every x holds $x \in D$ if and only if x is a submodule of V, then $D = \mathrm{Sub}(V)$.

(43) $x \in \mathrm{Sub}(V)$ if and only if x is a submodule of V.

(44) $V \in \mathrm{Sub}(V)$.

Let us consider R, V, W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def.4) $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

One can prove the following two propositions:

(46)[2] If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) V is the direct sum of $\mathbf{0}_V$ and Ω_V and V is the direct sum of Ω_V and $\mathbf{0}_V$.

In the sequel C_1 will denote a coset of W_1 and C_2 will denote a coset of W_2. Next we state several propositions:

(48) If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(49) V is the direct sum of W_1 and W_2 if and only if for every C_1, C_2 there exists v such that $C_1 \cap C_2 = \{v\}$.

(50) $W_1 + W_2 = V$ if and only if for every v there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(51) If V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(52) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

In the sequel t will be an element of [: the carrier of the carrier of V, the carrier of the carrier of V :]. Let us consider R, V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v \lhd (W_1, W_2)$ yielding an

[2]The proposition (45) was either repeated or obvious.

element of [: the carrier of the carrier of V, the carrier of the carrier of V :] is defined as follows:

(Def.5) $v = (v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}}$ and $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$ and $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

The following propositions are true:

(53) If V is the direct sum of W_1 and W_2 and $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W_1$ and $t_{\mathbf{2}} \in W_2$, then $t = v \lhd (W_1, W_2)$.

(54) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}} = v$.

(55) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$.

(56) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

(57) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} = (v \lhd (W_2, W_1))_{\mathbf{2}}$.

(58) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} = (v \lhd (W_2, W_1))_{\mathbf{1}}$.

In the sequel A_1, A_2 will denote elements of $\mathrm{Sub}(V)$. Let us consider R, V. The functor SubJoin V yields a binary operation on $\mathrm{Sub}(V)$ and is defined as follows:

(Def.6) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, V)(A_1, A_2) = W_1 + W_2$.

Let us consider R, V. The functor SubMeet V yielding a binary operation on $\mathrm{Sub}(V)$ is defined as follows:

(Def.7) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, V)(A_1, A_2) = W_1 \cap W_2$.

In the sequel o is a binary operation on $\mathrm{Sub}(V)$. Next we state several propositions:

(59) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubJoin}\, V(A_1, A_2) = W_1 + W_2$.

(60) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 + W_2$, then $o = \mathrm{SubJoin}\, V$.

(61) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubMeet}\, V(A_1, A_2) = W_1 \cap W_2$.

(62) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 \cap W_2$, then $o = \mathrm{SubMeet}\, V$.

(63) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lattice.

(64) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lower bound lattice.

(65) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is an upper bound lattice.

(66) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a bound lattice.

(67) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a modular lattice.

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