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Linear Combinations in Left Module over Associative Ring [1]

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Summary. Notion of linear combination of vectors in Left Module over Associative Ring, defined as a function from the carrier of Left Module over Associative Ring to the carrier of this Ring. The following operations are included: addition, subtraction of combinations and multiplication of a combination by a scalar of the Ring. Following it, the sum of a finite set of vectors and the sum of linear combinations is defined. Many theorems are proved. This article originated as a generalization of the article [19].

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The articles [22], [7], [5], [3], [6], [8], [21], [17], [15], [16], [2], [4], [18], [20], [1], [9], [10], [11], [13], [12], and [14] provide the terminology and notation for this paper. For simplicity we follow a convention: R will be an associative ring, V will be a left module over R, a, b will be scalars of R, x will be arbitrary, i will be a natural number, u, v, v_1, v_2, v_3 will be vectors of V, F, G will be finite sequences of elements of the carrier of the carrier of V, A, B will be subsets of V, and f will be a function from the carrier of the carrier of V into the carrier of R. Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

Let us consider R, V. A subset of V is said to be a finite subset of V if:

(Def.1) it is finite.

In the sequel S, T denote finite subsets of V. Let us consider R, V, S, T. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let us consider R, V. The functor 0_V yields a finite subset of V and is defined as follows:

(Def.2) $0_V = \emptyset$.

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One can prove the following proposition

(2)[2] $0_V = \emptyset$.

Let us consider R, V, T. The functor $\sum T$ yields a vector of V and is defined as follows:

(Def.3) there exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

One can prove the following two propositions:

(3) There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

(4) If $\operatorname{rng} F = T$ and F is one-to-one and $v = \sum F$, then $v = \sum T$.

Let us consider R, V, v. Then $\{v\}$ is a finite subset of V.

Let us consider R, V, v_1, v_2. Then $\{v_1, v_2\}$ is a finite subset of V.

Let us consider R, V, v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

We now state a number of propositions:

(5) $\sum(0_V) = \Theta_V$.

(6) $\sum\{v\} = v$.

(7) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(8) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = v_1 + v_2 + v_3$.

(9) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(10) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(11) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(12) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(13) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(14) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(15) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider R, V. An element of $(\text{the carrier of } R)^{\text{the carrier of the carrier of } V}$ is called a linear combination of V if:

(Def.4) there exists T such that for every v such that $v \notin T$ holds $\mathrm{it}(v) = 0_R$.

In the sequel K, L, L_1, L_2, L_3 are linear combinations of V. We now state the proposition

(16) There exists T such that for every v such that $v \notin T$ holds $L(v) = 0_R$.

In the sequel E is an element of $(\text{the carrier of } R)^{\text{the carrier of the carrier of } V}$. Next we state the proposition

(17) If there exists T such that for every v such that $v \notin T$ holds $E(v) = 0_R$, then E is a linear combination of V.

Let us consider R, V, L. The functor $\operatorname{support} L$ yields a finite subset of V and is defined as follows:

(Def.5) $\operatorname{support} L = \{v : L(v) \neq 0_R\}$.

The following propositions are true:

[2]The proposition (1) was either repeated or obvious.

(18) support $L = \{v : L(v) \neq 0_R\}$.

(19) $x \in \operatorname{support} L$ if and only if there exists v such that $x = v$ and $L(v) \neq 0_R$.

(20) $L(v) = 0_R$ if and only if $v \notin \operatorname{support} L$.

Let us consider R, V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yielding a linear combination of V is defined by:

(Def.6) support $\mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

We now state two propositions:

(21) $L = \mathbf{0}_{\mathrm{LC}_V}$ if and only if support $L = \emptyset$.

(22) $\mathbf{0}_{\mathrm{LC}_V}(v) = 0_R$.

Let us consider R, V, A. A linear combination of V is called a linear combination of A if:

(Def.7) support it $\subseteq A$.

We now state the proposition

(23) If support $L \subseteq A$, then L is a linear combination of A.

In the sequel l will denote a linear combination of A. We now state several propositions:

(24) support $l \subseteq A$.

(25) If $A \subseteq B$, then l is a linear combination of B.

(26) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(27) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(28) L is a linear combination of support L.

Let us consider R, V, F, f. The functor fF yields a finite sequence of elements of the carrier of the carrier of V and is defined by:

(Def.8) $\operatorname{len}(fF) = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom}(fF)$ holds $(fF)(i) = f(\pi_i F) \cdot \pi_i F$.

We now state several propositions:

(29) $\operatorname{len}(fF) = \operatorname{len} F$.

(30) For every i such that $i \in \operatorname{dom}(fF)$ holds $(fF)(i) = f(\pi_i F) \cdot \pi_i F$.

(31) If $\operatorname{len} G = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = f(\pi_i F) \cdot \pi_i F$, then $G = fF$.

(32) If $i \in \operatorname{dom} F$ and $v = F(i)$, then $(fF)(i) = f(v) \cdot v$.

(33) $f\varepsilon_{\text{the carrier of the carrier of } V} = \varepsilon_{\text{the carrier of the carrier of } V}$.

(34) $f\langle v \rangle = \langle f(v) \cdot v \rangle$.

(35) $f\langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(36) $f\langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

(37) $f(F \frown G) = (fF) \frown (fG)$.

Let us consider R, V, L. The functor $\sum L$ yields a vector of V and is defined as follows:

(Def.9) there exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L F)$.

The following propositions are true:

(38) There exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L F)$.

(39) If F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $u = \sum(L F)$, then $u = \sum L$.

(40) If $0_R \neq 1_R$, then $A \neq \emptyset$ and A is linearly closed if and only if for every l holds $\sum l \in A$.

(41) $\sum \mathbf{0}_{\mathrm{LC}_V} = \Theta_V$.

(42) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $\sum l = \Theta_V$.

(43) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(44) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(45) If $\operatorname{support} L = \emptyset$, then $\sum L = \Theta_V$.

(46) If $\operatorname{support} L = \{v\}$, then $\sum L = L(v) \cdot v$.

(47) If $\operatorname{support} L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let us consider R, V, L_1, L_2. Let us note that one can characterize the predicate $L_1 = L_2$ by the following (equivalent) condition:

(Def.10) for every v holds $L_1(v) = L_2(v)$.

Next we state the proposition

(48) If for every v holds $L_1(v) = L_2(v)$, then $L_1 = L_2$.

Let us consider R, V, L_1, L_2. The functor $L_1 + L_2$ yielding a linear combination of V is defined by:

(Def.11) for every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

The following propositions are true:

(49) If for every v holds $L(v) = L_1(v) + L_2(v)$, then $L = L_1 + L_2$.

(50) $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

(51) $\operatorname{support}(L_1 + L_2) \subseteq \operatorname{support} L_1 \cup \operatorname{support} L_2$.

(52) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 + L_2$ is a linear combination of A.

(53) For every commutative ring R and for every left module V over R and for all linear combinations L_1, L_2 of V holds $L_1 + L_2 = L_2 + L_1$.

(54) $L_1 + (L_2 + L_3) = L_1 + L_2 + L_3$.

(55) For every commutative ring R and for every left module V over R and for every linear combination L of V holds $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider R, V, a, L. The functor $a \cdot L$ yielding a linear combination of V is defined as follows:

(Def.12) for every v holds $(a \cdot L)(v) = a \cdot L(v)$.

One can prove the following propositions:

(56) If for every v holds $K(v) = a \cdot L(v)$, then $K = a \cdot L$.

(57) $(a \cdot L)(v) = a \cdot L(v)$.

(58) $\mathrm{support}(a \cdot L) \subseteq \mathrm{support}\, L$.

In the sequel R_1 denotes an integral domain, V_1 denotes a left module over R_1, L_4 denotes a linear combination of V_1, and a_1 denotes a scalar of R_1. Next we state several propositions:

(59) If $a_1 \neq 0_{R_1}$, then $\mathrm{support}(a_1 \cdot L_4) = \mathrm{support}\, L_4$.

(60) $0_R \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(61) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(62) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(63) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(64) $a \cdot (b \cdot L) = a \cdot b \cdot L$.

(65) $(1_R) \cdot L = L$.

Let us consider R, V, L. The functor $-L$ yields a linear combination of V and is defined as follows:

(Def.13) $-L = (-1_R) \cdot L$.

One can prove the following propositions:

(66) $-L = (-1_R) \cdot L$.

(67) $(-L)(v) = -L(v)$.

(68) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(69) $\mathrm{support} -L = \mathrm{support}\, L$.

(70) If L is a linear combination of A, then $-L$ is a linear combination of A.

(71) $- - L = L$.

Let us consider R, V, L_1, L_2. The functor $L_1 - L_2$ yields a linear combination of V and is defined by:

(Def.14) $L_1 - L_2 = L_1 + -L_2$.

One can prove the following propositions:

(72) $L_1 - L_2 = L_1 + -L_2$.

(73) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(74) $\mathrm{support}(L_1 - L_2) \subseteq \mathrm{support}\, L_1 \cup \mathrm{support}\, L_2$.

(75) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 - L_2$ is a linear combination of A.

(76) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

(77) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

For simplicity we adopt the following convention: R will be an integral domain, V will be a left module over R, L, L_1, L_2 will be linear combinations of V, and a will be a scalar of R. We now state three propositions:

(78) $\sum(a \cdot L) = a \cdot \sum L$.

(79) $\sum -L = -\sum L$.

(80) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

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