

N-Tuples and Cartesian Products for $n=5$ ¹

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Summary. This article defines ordered n -tuples, projections and Cartesian products for $n=5$. We prove many theorems concerning the basic properties of the n -tuples and Cartesian products that may be utilized in several further, more challenging applications. A few of these theorems are a straightforward consequence of the regularity axiom. The article originated as an upgrade of the article [5].

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The notation and terminology used in this paper are introduced in the following articles: [4], [3], [6], [2], [1], and [5]. For simplicity we follow a convention: v will be arbitrary, x_1, x_2, x_3, x_4, x_5 will be arbitrary, y_1, y_2, y_3, y_4, y_5 will be arbitrary, z will be arbitrary, $X, X_1, X_2, X_3, X_4, X_5$ will denote sets, $Y, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ will denote sets, Z will denote a set, x_6 will denote an element of X_1 , x_7 will denote an element of X_2 , x_8 will denote an element of X_3 , and x_9 will denote an element of X_4 . We now state two propositions:

- (1) If $X \neq \emptyset$, then there exists Y such that $Y \in X$ and for all $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y$ holds Y_1 misses X .
- (2) If $X \neq \emptyset$, then there exists Y such that $Y \in X$ and for all $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y_7$ and $Y_7 \in Y$ holds Y_1 misses X .

Let us consider x_1, x_2, x_3, x_4, x_5 . The functor $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is defined as follows:

(Def.1) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2, x_3, x_4 \rangle, x_5 \rangle$.

One can prove the following propositions:

- (3) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle \langle x_1, x_2 \rangle, x_3 \rangle, x_4 \rangle, x_5 \rangle$.
- (4) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2, x_3, x_4 \rangle, x_5 \rangle$.

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- (5) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2, x_3 \rangle, x_4, x_5 \rangle$.
 (6) $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle \langle x_1, x_2 \rangle, x_3, x_4, x_5 \rangle$.
 (7) If $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle y_1, y_2, y_3, y_4, y_5 \rangle$, then $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$ and $x_5 = y_5$.
 (8) If $X \neq \emptyset$, then there exists v such that $v \in X$ and for no x_1, x_2, x_3, x_4, x_5 holds $x_1 \in X$ or $x_2 \in X$ but $v = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

Let us consider X_1, X_2, X_3, X_4, X_5 . The functor $[\![X_1, X_2, X_3, X_4, X_5]\!]$ yields a set and is defined as follows:

$$\text{(Def.2)} \quad [\![X_1, X_2, X_3, X_4, X_5]\!] = [\![[\![X_1, X_2, X_3, X_4]\!] , X_5]\!]$$

The following propositions are true:

- (9) $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![[\![[\![X_1, X_2]\!] , X_3]\!] , X_4]\!] , X_5]\!]$.
 (10) $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![[\![X_1, X_2, X_3, X_4]\!] , X_5]\!]$.
 (11) $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![[\![X_1, X_2, X_3]\!] , X_4, X_5]\!]$.
 (12) $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![[\![X_1, X_2]\!] , X_3, X_4, X_5]\!]$.
 (13) $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ if and only if $[\![X_1, X_2, X_3, X_4, X_5]\!] \neq \emptyset$.
 (14) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Then if $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![Y_1, Y_2, Y_3, Y_4, Y_5]\!]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$.
 (15) If $[\![X_1, X_2, X_3, X_4, X_5]\!] \neq \emptyset$ and $[\![X_1, X_2, X_3, X_4, X_5]\!] = [\![Y_1, Y_2, Y_3, Y_4, Y_5]\!]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$.
 (16) If $[\![X, X, X, X, X]\!] = [\![Y, Y, Y, Y, Y]\!]$, then $X = Y$.

In the sequel x_{10} will be an element of X_5 . We now state the proposition

- (17) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$, then for every element x of $[\![X_1, X_2, X_3, X_4, X_5]\!]$ there exist $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$.

We now define five new functors. Let us consider X_1, X_2, X_3, X_4, X_5 . Let us assume that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let x be an element of $[\![X_1, X_2, X_3, X_4, X_5]\!]$. The functor x_1 yields an element of X_1 and is defined as follows:

$$\text{(Def.3)} \quad \text{if } x = \langle x_1, x_2, x_3, x_4, x_5 \rangle, \text{ then } x_1 = x_1.$$

The functor x_2 yields an element of X_2 and is defined as follows:

$$\text{(Def.4)} \quad \text{if } x = \langle x_1, x_2, x_3, x_4, x_5 \rangle, \text{ then } x_2 = x_2.$$

The functor x_3 yielding an element of X_3 is defined as follows:

$$\text{(Def.5)} \quad \text{if } x = \langle x_1, x_2, x_3, x_4, x_5 \rangle, \text{ then } x_3 = x_3.$$

The functor x_4 yielding an element of X_4 is defined as follows:

$$\text{(Def.6)} \quad \text{if } x = \langle x_1, x_2, x_3, x_4, x_5 \rangle, \text{ then } x_4 = x_4.$$

The functor x_5 yields an element of X_5 and is defined by:

$$\text{(Def.7)} \quad \text{if } x = \langle x_1, x_2, x_3, x_4, x_5 \rangle, \text{ then } x_5 = x_5.$$

One can prove the following propositions:

- (18) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Then for every element x of $[X_1, X_2, X_3, X_4, X_5]$ and for all x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ holds $x_1 = x_1$ and $x_2 = x_2$ and $x_3 = x_3$ and $x_4 = x_4$ and $x_5 = x_5$.
- (19) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$, then for every element x of $[X_1, X_2, X_3, X_4, X_5]$ holds $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.
- (20) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Let x be an element of $[X_1, X_2, X_3, X_4, X_5]$. Then $x_1 = x$ **qua any₁₁₁₁₁** and $x_2 = x$ **qua any₁₁₁₂** and $x_3 = x$ **qua any₁₁₂** and $x_4 = x$ **qua any₁₂** and $x_5 = x$ **qua any₂**.
- (21) If $X_1 \subseteq [X_1, X_2, X_3, X_4, X_5]$ or $X_1 \subseteq [X_2, X_3, X_4, X_5, X_1]$ or $X_1 \subseteq [X_3, X_4, X_5, X_1, X_2]$ or $X_1 \subseteq [X_4, X_5, X_1, X_2, X_3]$ or $X_1 \subseteq [X_5, X_1, X_2, X_3, X_4]$, then $X_1 = \emptyset$.
- (22) If $[X_1, X_2, X_3, X_4, X_5]$ meets $[Y_1, Y_2, Y_3, Y_4, Y_5]$, then X_1 meets Y_1 and X_2 meets Y_2 and X_3 meets Y_3 and X_4 meets Y_4 and X_5 meets Y_5 .
- (23) $[\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}] = \{\langle x_1, x_2, x_3, x_4, x_5 \rangle\}$.

For simplicity we adopt the following rules: A_1 is a subset of X_1 , A_2 is a subset of X_2 , A_3 is a subset of X_3 , A_4 is a subset of X_4 , A_5 is a subset of X_5 , and x is an element of $[X_1, X_2, X_3, X_4, X_5]$. One can prove the following propositions:

- (24) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$. Then for all x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ holds $x_1 = x_1$ and $x_2 = x_2$ and $x_3 = x_3$ and $x_4 = x_4$ and $x_5 = x_5$.
- (25) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_1 = x_6$, then $y_1 = x_1$.
- (26) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_2 = x_7$, then $y_2 = x_2$.
- (27) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_3 = x_8$, then $y_3 = x_3$.
- (28) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_4 = x_9$, then $y_4 = x_4$.
- (29) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and for all $x_6, x_7, x_8, x_9, x_{10}$ such that $x = \langle x_6, x_7, x_8, x_9, x_{10} \rangle$ holds $y_5 = x_{10}$, then $y_5 = x_5$.
- (30) If $z \in [X_1, X_2, X_3, X_4, X_5]$, then there exist x_1, x_2, x_3, x_4, x_5 such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $z = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

- (31) $\langle x_1, x_2, x_3, x_4, x_5 \rangle \in [X_1, X_2, X_3, X_4, X_5]$ if and only if $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$.
- (32) If for every z holds $z \in Z$ if and only if there exist x_1, x_2, x_3, x_4, x_5 such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $z = \langle x_1, x_2, x_3, x_4, x_5 \rangle$, then $Z = [X_1, X_2, X_3, X_4, X_5]$.
- (33) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ and $Y_3 \neq \emptyset$ and $Y_4 \neq \emptyset$ and $Y_5 \neq \emptyset$. Let x be an element of $[X_1, X_2, X_3, X_4, X_5]$. Then for every element y of $[Y_1, Y_2, Y_3, Y_4, Y_5]$ such that $x = y$ holds $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$ and $x_5 = y_5$.
- (34) For every element x of $[X_1, X_2, X_3, X_4, X_5]$ such that $x \in [A_1, A_2, A_3, A_4, A_5]$ holds $x_1 \in A_1$ and $x_2 \in A_2$ and $x_3 \in A_3$ and $x_4 \in A_4$ and $x_5 \in A_5$.
- (35) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ and $X_3 \subseteq Y_3$ and $X_4 \subseteq Y_4$ and $X_5 \subseteq Y_5$, then $[X_1, X_2, X_3, X_4, X_5] \subseteq [Y_1, Y_2, Y_3, Y_4, Y_5]$.

Let us consider $X_1, X_2, X_3, X_4, X_5, A_1, A_2, A_3, A_4, A_5$. Then $[A_1, A_2, A_3, A_4, A_5]$ is a subset of $[X_1, X_2, X_3, X_4, X_5]$.

The following three propositions are true:

- (36) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, then for every element x_{11} of $[X_1, X_2]$ there exists an element x_6 of X_1 and there exists an element x_7 of X_2 such that $x_{11} = \langle x_6, x_7 \rangle$.
- (37) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$, then for every element x_{11} of $[X_1, X_2, X_3]$ there exist x_6, x_7, x_8 such that $x_{11} = \langle x_6, x_7, x_8 \rangle$.
- (38) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$, then for every element x_{11} of $[X_1, X_2, X_3, X_4]$ there exist x_6, x_7, x_8, x_9 such that $x_{11} = \langle x_6, x_7, x_8, x_9 \rangle$.

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