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On Pseudometric Spaces [1]

Adam Lecko
Technical University of Rzeszów

Mariusz Startek
Technical University of Rzeszów

Summary. We introduce the equivalence classes in a pseudometric space. Next we prove that the set of the equivalence classes forms the metric space with the special metric defined in the article.

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The terminology and notation used here have been introduced in the following articles: [9], [4], [13], [12], [10], [8], [2], [3], [1], [14], [7], [11], [5], and [6]. Let M be a metric structure, and let x, y be elements of the carrier of M. The predicate $x \approx y$ is defined by:

(Def.1) $\rho(x, y) = 0$.

Let M be a metric structure, and let x be an element of the carrier of M. The functor $x^{\square}$ yielding a subset of the carrier of M is defined as follows:

(Def.2) $x^{\square} = \{y : x \approx y\}$, where y ranges over elements of the carrier of M.

One can prove the following proposition

(2)[2] For every M being a metric structure and for every element x of the carrier of M holds $x^{\square} = \{y : x \approx y\}$, where y ranges over elements of the carrier of M.

Let M be a metric structure. A subset of the carrier of M is called a $\square$-equivalence class of M if:

(Def.3) there exists an element x of the carrier of M such that it $= x^{\square}$.

Next we state a number of propositions:

(4)[3] For every pseudo metric space M and for every element x of the carrier of M holds $x \approx x$.

(5) For every pseudo metric space M and for all elements x, y of the carrier of M such that $x \approx y$ holds $y \approx x$.

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[2]The proposition (1) was either repeated or obvious.
[3]The proposition (3) was either repeated or obvious.

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(6) For every pseudo metric space M and for all elements x, y, z of the carrier of M such that $x \approx y$ and $y \approx z$ holds $x \approx z$.

(7) For every pseudo metric space M and for all elements x, y of the carrier of M holds $y \in x^{\square}$ if and only if $y \approx x$.

(8) For every pseudo metric space M and for all elements x, p, q of the carrier of M such that $p \in x^{\square}$ and $q \in x^{\square}$ holds $p \approx q$.

(9) For every pseudo metric space M and for every element x of the carrier of M holds $x \in x^{\square}$.

(10) For every pseudo metric space M and for all elements x, y of the carrier of M holds $x \in y^{\square}$ if and only if $y \in x^{\square}$.

(11) For every pseudo metric space M and for all elements p, x, y of the carrier of M such that $p \in x^{\square}$ and $x \approx y$ holds $p \in y^{\square}$.

(12) For every pseudo metric space M and for all elements x, y of the carrier of M such that $y \in x^{\square}$ holds $x^{\square} = y^{\square}$.

(13) For every pseudo metric space M and for all elements x, y of the carrier of M holds $x^{\square} = y^{\square}$ if and only if $x \approx y$.

The following propositions are true:

(14) For every pseudo metric space M and for all elements x, y of the carrier of M holds $x^{\square} \cap y^{\square} \neq \emptyset$ if and only if $x \approx y$.

(15) For every pseudo metric space M and for every element x of the carrier of M holds $x^{\square}$ is a non-empty set.

(16) For every pseudo metric space M and for every $\square$-equivalence class V of M holds V is a non-empty set.

(17) For every pseudo metric space M and for all elements x, p, q of the carrier of M such that $p \in x^{\square}$ and $q \in x^{\square}$ holds $\rho(p, q) = 0$.

(18) For every metric space M and for all elements x, y of the carrier of M holds $x \approx y$ if and only if $x = y$.

(19) For every metric space M and for all elements x, y of the carrier of M holds $y \in x^{\square}$ if and only if $y = x$.

One can prove the following two propositions:

(20) For every metric space M and for every element x of the carrier of M holds $x^{\square} = \{x\}$.

(21) For every metric space M and for every subset V of the carrier of M holds V is a $\square$-equivalence class of M if and only if there exists an element x of the carrier of M such that $V = \{x\}$.

Let M be a metric structure. The functor $M^{\square}$ yields a non-empty set and is defined by:

(Def.4) $M^{\square} = \{s : \bigvee_x x^{\square} = s\}$, where s ranges over elements of $2^{\text{the carrier of } M}$, and x ranges over elements of the carrier of M.

One can prove the following proposition

(22) For every M being a metric structure holds $M^{\square} = \{s : \bigvee_x x^{\square} = s\}$, where s ranges over elements of $2^{\text{the carrier of } M}$, and x ranges over elements of the carrier of M.

In the sequel V is arbitrary. The following two propositions are true:

(23) For every M being a metric structure holds $V \in M^{\square}$ if and only if there exists an element x of the carrier of M such that $V = x^{\square}$.

(24) For every M being a metric structure and for every element x of the carrier of M holds $x^{\square} \in M^{\square}$.

We now state the proposition

(26)[4] For every M being a metric structure holds $V \in M^{\square}$ if and only if V is a $\square$-equivalence class of M.

We now state three propositions:

(27) For every metric space M and for every element x of the carrier of M holds $\{x\} \in M^{\square}$.

(28) For every metric space M holds $V \in M^{\square}$ if and only if there exists an element x of the carrier of M such that $V = \{x\}$.

(29) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ and for all elements p_1, p_2, q_1, q_2 of the carrier of M such that $p_1 \in V$ and $q_1 \in Q$ and $p_2 \in V$ and $q_2 \in Q$ holds $\rho(p_1, q_1) = \rho(p_2, q_2)$.

Let M be a pseudo metric space, and let V, Q be elements of $M^{\square}$, and let v be an element of $\mathbb{R}$. We say that the distance between V and Q is v if and only if:

(Def.5) for all elements p, q of the carrier of M such that $p \in V$ and $q \in Q$ holds $\rho(p, q) = v$.

We now state two propositions:

(31)[5] For every pseudo metric space M and for all elements V, Q of $M^{\square}$ and for every element v of $\mathbb{R}$ holds the distance between V and Q is v if and only if there exist elements p, q of the carrier of M such that $p \in V$ and $q \in Q$ and $\rho(p, q) = v$.

(32) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ and for every element v of $\mathbb{R}$ holds the distance between V and Q is v if and only if the distance between Q and V is v.

Let M be a pseudo metric space, and let V, Q be elements of $M^{\square}$. The functor $\rho^{\circ}(V, Q)$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.6) $\rho^{\circ}(V, Q) = \{v :$ the distance between V and Q is $v\}$, where v ranges over elements of $\mathbb{R}$.

The following two propositions are true:

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[4]The proposition (25) was either repeated or obvious.

[5]The proposition (30) was either repeated or obvious.

(33) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ holds $\rho^{\circ}(V,Q) = \{v :$ the distance between V and Q is $v\}$, where v ranges over elements of $\mathbb{R}$.

(34) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ and for every element v of $\mathbb{R}$ holds $v \in \rho^{\circ}(V,Q)$ if and only if the distance between V and Q is v.

Let M be a pseudo metric space, and let v be an element of $\mathbb{R}$. The functor $\rho_M^{\square\,-1}(v)$ yields a subset of $[\!: M^{\square}, M^{\square} :\!]$ and is defined as follows:

(Def.7) $\rho_M^{\square\,-1}(v) = \{W : \bigvee_{V,Q}[W = \langle V,Q \rangle \wedge$ the distance between V and Q is $v\,]\}$, where W ranges over elements of $[\!: M^{\square}, M^{\square} :\!]$, and V, Q range over elements of $M^{\square}$.

One can prove the following two propositions:

(35) For every pseudo metric space M and for every element v of $\mathbb{R}$ holds $\rho_M^{\square\,-1}(v) = \{W : \bigvee_{V,Q}[W = \langle V,Q \rangle \wedge$ the distance between V and Q is $v\,]\}$, where W ranges over elements of $[\!: M^{\square}, M^{\square} :\!]$, and V, Q range over elements of $M^{\square}$.

(36) For every pseudo metric space M and for every element v of $\mathbb{R}$ and for every element W of $[\!: M^{\square}, M^{\square} :\!]$ holds $W \in \rho_M^{\square\,-1}(v)$ if and only if there exist elements V, Q of $M^{\square}$ such that $W = \langle V,Q \rangle$ and the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\rho^{\circ}(M^{\square}, M^{\square})$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.8) $\rho^{\circ}(M^{\square}, M^{\square}) = \{v : \bigvee_{V,Q}$ the distance between V and Q is $v\}$, where v ranges over elements of $\mathbb{R}$, and V, Q range over elements of $M^{\square}$.

The following two propositions are true:

(37) For every pseudo metric space M holds $\rho^{\circ}(M^{\square}, M^{\square}) = \{v : \bigvee_{V,Q}$ the distance between V and Q is $v\}$, where v ranges over elements of $\mathbb{R}$, and V, Q range over elements of $M^{\square}$.

(38) For every pseudo metric space M and for every element v of $\mathbb{R}$ holds $v \in \rho^{\circ}(M^{\square}, M^{\square})$ if and only if there exist elements V, Q of $M^{\square}$ such that the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\mathrm{dom}_1\,\rho_M^{\square}$ yields a subset of $M^{\square}$ and is defined as follows:

(Def.9) $\mathrm{dom}_1\,\rho_M^{\square} = \{V : \bigvee_Q \bigvee_v$ the distance between V and Q is $v\}$, where V ranges over elements of $M^{\square}$, and Q ranges over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

We now state two propositions:

(39) For every pseudo metric space M holds $\mathrm{dom}_1\,\rho_M^{\square} = \{V : \bigvee_Q \bigvee_v$ the distance between V and Q is $v\}$, where V ranges over elements of $M^{\square}$, and Q ranges over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

(40) For every pseudo metric space M and for every element V of $M^{\square}$ holds $V \in \mathrm{dom}_1\, \rho_M^{\square}$ if and only if there exists an element Q of $M^{\square}$ and there exists an element v of $\mathbb{R}$ such that the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\mathrm{dom}_2\, \rho_M^{\square}$ yields a subset of $M^{\square}$ and is defined by:

(Def.10) $\mathrm{dom}_2\, \rho_M^{\square} = \{Q : \bigvee_V \bigvee_v$ the distance between V and Q is $v\ \}$, where Q ranges over elements of $M^{\square}$, and V ranges over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

One can prove the following two propositions:

(41) For every pseudo metric space M holds $\mathrm{dom}_2\, \rho_M^{\square} = \{Q : \bigvee_V \bigvee_v$ the distance between V and Q is $v\ \}$, where Q ranges over elements of $M^{\square}$, and V ranges over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

(42) For every pseudo metric space M and for every element Q of $M^{\square}$ holds $Q \in \mathrm{dom}_2\, \rho_M^{\square}$ if and only if there exists an element V of $M^{\square}$ and there exists an element v of $\mathbb{R}$ such that the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\mathrm{dom}\, \rho_M^{\square}$ yielding a subset of $[\!:M^{\square}, M^{\square}:\!]$ is defined as follows:

(Def.11) $\mathrm{dom}\, \rho_M^{\square} = \{V_1 : \bigvee_{V,Q} \bigvee_v [V_1 = \langle V, Q \rangle \wedge$ the distance between V and Q is $v\ ]\}$, where V_1 ranges over elements of $[\!:M^{\square}, M^{\square}:\!]$, and V, Q range over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

We now state two propositions:

(43) For every pseudo metric space M holds $\mathrm{dom}\, \rho_M^{\square} = \{V_1 : \bigvee_{V,Q} \bigvee_v [V_1 = \langle V, Q \rangle \wedge$ the distance between V and Q is $v\ ]\}$, where V_1 ranges over elements of $[\!:M^{\square}, M^{\square}:\!]$, and V, Q range over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

(44) For every pseudo metric space M and for every element V_1 of $[\!:M^{\square}, M^{\square}:\!]$ holds $V_1 \in \mathrm{dom}\, \rho_M^{\square}$ if and only if there exist elements V, Q of $M^{\square}$ and there exists an element v of $\mathbb{R}$ such that $V_1 = \langle V, Q \rangle$ and the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\mathrm{graph}\, \rho_M^{\square}$ yielding a subset of $[\!:M^{\square}, M^{\square}, \mathbb{R}:\!]$ is defined by:

(Def.12) $\mathrm{graph}\, \rho_M^{\square} = \{V_2 : \bigvee_{V,Q} \bigvee_v [V_2 = \langle V, Q, v \rangle \wedge$ the distance between V and Q is $v\ ]\}$, where V_2 ranges over elements of $[\!:M^{\square}, M^{\square}, \mathbb{R}:\!]$, and V, Q range over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

The following propositions are true:

(45) For every pseudo metric space M holds $\mathrm{graph}\, \rho_M^{\square} = \{V_2 : \bigvee_{V,Q} \bigvee_v [V_2 = \langle V, Q, v \rangle \wedge$ the distance between V and Q is $v\ ]\}$, where V_2 ranges over elements of $[\!:M^{\square}, M^{\square}, \mathbb{R}:\!]$, and V, Q range over elements of $M^{\square}$, and v ranges over elements of $\mathbb{R}$.

(46) For every pseudo metric space M and for every element V_2 of $[\!:M^{\square}, M^{\square}, \mathbb{R}:\!]$ holds $V_2 \in \mathrm{graph}\, \rho_M^{\square}$ if and only if there exist elements V, Q of

$M^{\square}$ and there exists an element v of $\mathbb{R}$ such that $V_2 = \langle V, Q, v \rangle$ and the distance between V and Q is v.

(47) For every pseudo metric space M holds $\mathrm{dom}_1\, \rho^{\square}_M = \mathrm{dom}_2\, \rho^{\square}_M$.

(48) For every pseudo metric space M holds graph $\rho^{\square}_M \subseteq [\!:\mathrm{dom}_1\, \rho^{\square}_M, \mathrm{dom}_2\, \rho^{\square}_M, \rho^{\circ}(M^{\square}, M^{\square})\,:\!]$.

(49) Let M be a pseudo metric space. Then for all elements V, Q of $M^{\square}$ and for all elements p_1, q_1, p_2, q_2 of the carrier of M and for all elements v_1, v_2 of $\mathbb{R}$ such that $p_1 \in V$ and $q_1 \in Q$ and $\rho(p_1, q_1) = v_1$ and $p_2 \in V$ and $q_2 \in Q$ and $\rho(p_2, q_2) = v_2$ holds $v_1 = v_2$.

The following two propositions are true:

(50) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ and for all elements v_1, v_2 of $\mathbb{R}$ such that the distance between V and Q is v_1 and the distance between V and Q is v_2 holds $v_1 = v_2$.

(52)[6] For every pseudo metric space M and for every elements V, Q of $M^{\square}$ there exists an element v of $\mathbb{R}$ such that the distance between V and Q is v.

Let M be a pseudo metric space. The functor $\rho^{\square}_M$ yielding a function from $[\!:M^{\square}, M^{\square}:\!]$ into $\mathbb{R}$ is defined as follows:

(Def.13) for all elements V, Q of $M^{\square}$ and for all elements p, q of the carrier of M such that $p \in V$ and $q \in Q$ holds $\rho^{\square}_M(V, Q) = \rho(p, q)$.

One can prove the following propositions:

(53) For every pseudo metric space M and for every function F from $[\!:M^{\square}, M^{\square}:\!]$ into $\mathbb{R}$ holds $F = \rho^{\square}_M$ if and only if for all elements V, Q of $M^{\square}$ and for all elements p, q of the carrier of M such that $p \in V$ and $q \in Q$ holds $F(V, Q) = \rho(p, q)$.

(54) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ holds $\rho^{\square}_M(V, Q) = 0$ if and only if $V = Q$.

(55) For every pseudo metric space M and for all elements V, Q of $M^{\square}$ holds $\rho^{\square}_M(V, Q) = \rho^{\square}_M(Q, V)$.

(56) For every pseudo metric space M and for all elements V, Q, W of $M^{\square}$ holds $\rho^{\square}_M(V, W) \le \rho^{\square}_M(V, Q) + \rho^{\square}_M(Q, W)$.

Let M be a pseudo metric space. The functor $M_{/\square}$ yields a metric space and is defined as follows:

(Def.14) $M_{/\square} = \langle M^{\square}, \rho^{\square}_M \rangle$.

We now state the proposition

(57) For every pseudo metric space M holds $M_{/\square} = \langle M^{\square}, \rho^{\square}_M \rangle$.

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