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Metrics in Cartesian Product [1]

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Summary. A continuation of the paper [8]. It deals with the method of creation of the distance in the Cartesian product of metric spaces. The distance of two points belonging to the Cartesian product of metric spaces has been defined as the sum of distances of appriopriate coordinates (or projections) of these points. It is shown that the product of metric spaces with such a distance is a metric space.

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The articles [7], [12], [4], [5], [2], [6], [1], [9], [3], [8], [11], and [10] provide the notation and terminology for this paper. We follow the rules: X, Y will denote metric spaces, x_1, y_1, z_1 will denote elements of the carrier of X, and x_2, y_2, z_2 will denote elements of the carrier of Y. The scheme *LambdaMCART* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, and a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$ and states that:

there exists a function f from $[\![[\![\mathcal{A}, \mathcal{B}]\!], [\![\mathcal{A}, \mathcal{B}]\!]]\!]$ into $\mathcal{C}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x, y of $[\![\mathcal{A}, \mathcal{B}]\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $f(\langle x, y \rangle) = \mathcal{F}(x_1, y_1, x_2, y_2)$
for all values of the parameters.

Let us consider X, Y. The functor $\rho^{X \times Y}$ yielding a function from $[\![[\![$ the carrier of X, the carrier of $Y]\!], [\![$ the carrier of X, the carrier of $Y]\!]]\!]$ into $\mathbb{R}$ is defined by:

(Def.1) for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of $[\![$ the carrier of X, the carrier of $Y]\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{X \times Y}(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2)$.

The following proposition is true

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(1) Let X be a metric space. Let Y be a metric space. Let F be a function from [: [: the carrier of X, the carrier of Y :], [: the carrier of X, the carrier of Y :] :] into $\mathbb{R}$. Then $F = \rho^{X \times Y}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of [: the carrier of X, the carrier of Y :] such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $F(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2)$.

One can prove the following proposition

(2) For all elements a, b of $\mathbb{R}$ such that $a + b = 0$ and $0 \leq a$ and $0 \leq b$ holds $a = 0$ and $b = 0$.

We now state four propositions:

(3) For every metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = 0$ if and only if $a = b$.

(5)[2] For all elements x, y of [: the carrier of X, the carrier of Y :] such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{X \times Y}(x, y) = 0$ if and only if $x = y$.

(6) For all elements x, y of [: the carrier of X, the carrier of Y :] such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{X \times Y}(x, y) = \rho^{X \times Y}(y, x)$.

(7) For all elements x, y, z of [: the carrier of X, the carrier of Y :] such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$ holds $\rho^{X \times Y}(x, z) \leq \rho^{X \times Y}(x, y) + \rho^{X \times Y}(y, z)$.

Let us consider X, Y, and let x, y be elements of [: the carrier of X, the carrier of Y :]. The functor $\rho(x, y)$ yielding a real number is defined as follows:

(Def.2) $\rho(x, y) = \rho^{X \times Y}(x, y)$.

We now state the proposition

(8) For all elements x, y of [: the carrier of X, the carrier of Y :] holds $\rho(x, y) = \rho^{X \times Y}(x, y)$.

Let X, Y be metric spaces. The functor [: X, Y :] yields a metric space and is defined as follows:

(Def.3) [: X, Y :] $= \langle$[: the carrier of X, the carrier of Y :]$, \rho^{X \times Y}\rangle$.

One can prove the following proposition

(9) For every metric space X and for every metric space Y holds $\langle$[: the carrier of X, the carrier of Y :]$, \rho^{X \times Y}\rangle$ is a metric space.

In the sequel Z will denote a metric space and x_3, y_3, z_3 will denote elements of the carrier of Z. The scheme *LambdaMCART1* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a non-empty set $\mathcal{D}$, and a 6-ary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$ and states that:

there exists a function f from [: [: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$:], [: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$:] :] into $\mathcal{D}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x_3, y_3 of $\mathcal{C}$ and for all elements x, y of [: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$:] such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $f(\langle x, y \rangle) = \mathcal{F}(x_1, y_1, x_2, y_2, x_3, y_3)$
for all values of the parameters.

[2] The proposition (4) was either repeated or obvious.

Let us consider X, Y, Z. The functor $\rho^{X \times Y \times Z}$ yielding a function from [: [: the carrier of X, the carrier of Y, the carrier of Z :], [: the carrier of X, the carrier of Y, the carrier of Z :] :] into $\mathbb{R}$ is defined by:

(Def.4) Let x_1, y_1 be elements of the carrier of X. Let x_2, y_2 be elements of the carrier of Y. Then for all elements x_3, y_3 of the carrier of Z and for all elements x, y of [: the carrier of X, the carrier of Y, the carrier of Z :] such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{X \times Y \times Z}(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2) + \rho(x_3, y_3)$.

Next we state four propositions:

(10) Let X be a metric space. Let Y be a metric space. Let Z be a metric space. Let F be a function from [: [: the carrier of X, the carrier of Y, the carrier of Z :], [: the carrier of X, the carrier of Y, the carrier of Z :] :] into $\mathbb{R}$. Then $F = \rho^{X \times Y \times Z}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x_3, y_3 of the carrier of Z and for all elements x, y of [: the carrier of X, the carrier of Y, the carrier of Z :] such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $F(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2) + \rho(x_3, y_3)$.

(12)[3] For all elements x, y of [: the carrier of X, the carrier of Y, the carrier of Z :] such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{X \times Y \times Z}(x, y) = 0$ if and only if $x = y$.

(13) For all elements x, y of [: the carrier of X, the carrier of Y, the carrier of Z :] such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{X \times Y \times Z}(x, y) = \rho^{X \times Y \times Z}(y, x)$.

(14) Let x, y, z be elements of [: the carrier of X, the carrier of Y, the carrier of Z :]. Then if $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$, then $\rho^{X \times Y \times Z}(x, z) \leq \rho^{X \times Y \times Z}(x, y) + \rho^{X \times Y \times Z}(y, z)$.

Let X, Y, Z be metric spaces. The functor [: X, Y, Z :] yields a metric space and is defined by:

(Def.5) [: X, Y, Z :] $= \langle$[: the carrier of X, the carrier of Y, the carrier of Z :], $\rho^{X \times Y \times Z} \rangle$.

Let us consider X, Y, Z, and let x, y be elements of [: the carrier of X, the carrier of Y, the carrier of Z :]. The functor $\rho(x, y)$ yielding a real number is defined by:

(Def.6) $\rho(x, y) = \rho^{X \times Y \times Z}(x, y)$.

The following propositions are true:

(15) For all elements x, y of [: the carrier of X, the carrier of Y, the carrier of Z :] holds $\rho(x, y) = \rho^{X \times Y \times Z}(x, y)$.

(16) For every metric space X and for every metric space Y and for every metric space Z holds $\langle$[: the carrier of X, the carrier of Y, the carrier of Z :], $\rho^{X \times Y \times Z} \rangle$ is a metric space.

[3]The proposition (11) was either repeated or obvious.

In the sequel W is a metric space and x_4, y_4, z_4 are elements of the carrier of W. The scheme *LambdaMCART2* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty set $\mathcal{C}$, a non-empty set $\mathcal{D}$, a non-empty set $\mathcal{E}$, and a 8-ary functor $\mathcal{F}$ yielding an element of $\mathcal{E}$ and states that:

there exists a function f from $[\![[\![\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]\!], [\![\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]\!]]\!]$ into $\mathcal{E}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x_3, y_3 of $\mathcal{C}$ and for all elements x_4, y_4 of $\mathcal{D}$ and for all elements x, y of $[\![\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]\!]$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ holds $f(\langle x, y \rangle) = \mathcal{F}(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$
for all values of the parameters.

Let us consider X, Y, Z, W. The functor $\rho^{X \times Y \times Z \times W}$ yielding a function from $[\![[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$, $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]]\!]$ into $\mathbb{R}$ is defined as follows:

(Def.7) Let x_1, y_1 be elements of the carrier of X. Let x_2, y_2 be elements of the carrier of Y. Let x_3, y_3 be elements of the carrier of Z. Let x_4, y_4 be elements of the carrier of W. Then for all elements x, y of $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ holds $\rho^{X \times Y \times Z \times W}(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2) + (\rho(x_3, y_3) + \rho(x_4, y_4))$.

The following propositions are true:

(17) Let X be a metric space. Let Y be a metric space. Let Z be a metric space. Let W be a metric space. Let F be a function from $[\![[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$, $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]]\!]$ into $\mathbb{R}$. Then $F = \rho^{X \times Y \times Z \times W}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x_3, y_3 of the carrier of Z and for all elements x_4, y_4 of the carrier of W and for all elements x, y of $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ holds $F(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2) + (\rho(x_3, y_3) + \rho(x_4, y_4))$.

(19)[4] For all elements x, y of $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ holds $\rho^{X \times Y \times Z \times W}(x, y) = 0$ if and only if $x = y$.

(20) For all elements x, y of $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ holds $\rho^{X \times Y \times Z \times W}(x, y) = \rho^{X \times Y \times Z \times W}(y, x)$.

(21) Let x, y, z be elements of $[\![$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W]\!]$. Then if $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ and $z = \langle z_1, z_2, z_3, z_4 \rangle$, then $\rho^{X \times Y \times Z \times W}(x, z) \leq \rho^{X \times Y \times Z \times W}(x, y) + \rho^{X \times Y \times Z \times W}(y, z)$.

[4] The proposition (18) was either repeated or obvious.

Let X, Y, Z, W be metric spaces. The functor $[: X, Y, Z, W :]$ yielding a metric space is defined as follows:

(Def.8) $[: X, Y, Z, W :] = \langle [:$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W :], \rho^{X \times Y \times Z \times W} \rangle$.

Let us consider X, Y, Z, W, and let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W :]$. The functor $\rho(x, y)$ yields a real number and is defined by:

(Def.9) $\rho(x, y) = \rho^{X \times Y \times Z \times W}(x, y)$.

One can prove the following propositions:

(22) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W :]$ holds $\rho(x, y) = \rho^{X \times Y \times Z \times W}(x, y)$.

(23) For every metric space X and for every metric space Y and for every metric space Z and for every metric space W holds $\langle [:$ the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W :], \rho^{X \times Y \times Z \times W} \rangle$ is a metric space.

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