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Ordered Rings - Part I

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Summary. This series of papers is devoted to the notion of the ordered ring, and one of its most important cases: the notion of ordered field. It follows the results of [5]. The idea of the notion of order in the ring is based on that of positive cone i.e. the set of positive elements. Positive cone has to contain at least squares of all elements, and has to be closed under sum and product. Therefore the key notions of this theory are that of square, sum of squares, product of squares, etc. and finally elements generated from squares by means of sums and products. Part I contains definitions of all those key notions and inclusions between them.

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The papers [1], [2], [6], [3], and [4] provide the notation and terminology for this paper. For simplicity we adopt the following convention: i, j, k, n will be natural numbers, R will be a field structure, x, y will be scalars of R, and f will be a finite sequence of elements of the carrier of R. Let us consider R, f, k. Let us assume that $0 \neq k$ and $k \leq \operatorname{len} f$. The functor $f \circ k$ yields a scalar of R and is defined by:

(Def.1) $f \circ k = f(k)$.

Let us consider R, x. The functor $x^{\mathbf{2}}$ yields a scalar of R and is defined as follows:

(Def.2) $x^{\mathbf{2}} = x \cdot x$.

Let us consider R, x. We say that x is a square if and only if:

(Def.3) there exists a scalar y of R such that $x = y^{\mathbf{2}}$.

Let us consider R, f. We say that f is a sequence of sums of squares if and only if:

(Def.4) $\operatorname{len} f \neq 0$ and $f \circ 1$ is a square and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a square and $f \circ (n+1) = f \circ n + y$.

Let us consider R, x. We say that x is a sum of squares if and only if:

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(Def.5) there exists f such that f is a sequence of sums of squares and $x = f \circ \operatorname{len} f$.

Let us consider R, f. We say that f is a sequence of products of squares if and only if:

(Def.6) $\operatorname{len} f \neq 0$ and $f \circ 1$ is a square and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a square and $f \circ (n+1) = f \circ n \cdot y$.

Let us consider R, x. We say that x is a product of squares if and only if:

(Def.7) there exists f such that f is a sequence of products of squares and $x = f \circ \operatorname{len} f$.

Let us consider R, f. We say that f is a sequence of sums of products of squares if and only if:

(Def.8) $\operatorname{len} f \neq 0$ and $f \circ 1$ is a product of squares and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a product of squares and $f \circ (n+1) = f \circ n + y$.

Let us consider R, x. We say that x is a sum of products of squares if and only if:

(Def.9) there exists f such that f is a sequence of sums of products of squares and $x = f \circ \operatorname{len} f$.

Let us consider R, f. We say that f is a sequence of amalgams of squares if and only if:

(Def.10) (i) $\operatorname{len} f \neq 0$,

(ii) for every n such that $n \neq 0$ and $n \leq \operatorname{len} f$ holds $f \circ n$ is a product of squares or there exist i, j such that $f \circ n = f \circ i \cdot f \circ j$ and $i \neq 0$ and $i < n$ and $j \neq 0$ and $j < n$.

Let us consider R, x. We say that x is a amalgam of squares if and only if:

(Def.11) there exists f such that f is a sequence of amalgams of squares and $x = f \circ \operatorname{len} f$.

Let us consider R, f. We say that f is a sequence of sums of amalgams of squares if and only if:

(Def.12) $\operatorname{len} f \neq 0$ and $f \circ 1$ is a amalgam of squares and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a amalgam of squares and $f \circ (n+1) = f \circ n + y$.

Let us consider R, x. We say that x is a sum of amalgams of squares if and only if:

(Def.13) there exists f such that f is a sequence of sums of amalgams of squares and $x = f \circ \operatorname{len} f$.

Let us consider R, f. We say that f is a generation from squares if and only if:

(Def.14) (i) $\operatorname{len} f \neq 0$,

(ii) for every n such that $n \neq 0$ and $n \leq \operatorname{len} f$ holds $f \circ n$ is a amalgam of squares or there exist i, j such that $f \circ n = f \circ i \cdot f \circ j$ or $f \circ n = f \circ i + f \circ j$ but $i \neq 0$ and $i < n$ and $j \neq 0$ and $j < n$.

Let us consider R, x. We say that x is generated from squares if and only if:

(Def.15) there exists f such that f is a generation from squares and $x = f \circ \operatorname{len} f$.

The following propositions are true:

(1) If x is a square, then x is a sum of squares and x is a product of squares and x is a sum of products of squares and x is a amalgam of squares and x is a sum of amalgams of squares and x is generated from squares.

(2) If x is a sum of squares, then x is a sum of products of squares and x is a sum of amalgams of squares and x is generated from squares.

(3) If x is a product of squares, then x is a sum of products of squares and x is a amalgam of squares and x is a sum of amalgams of squares and x is generated from squares.

(4) If x is a sum of products of squares, then x is a sum of amalgams of squares and x is generated from squares.

(5) If x is a amalgam of squares, then x is a sum of amalgams of squares and x is generated from squares.

(6) If x is a sum of amalgams of squares, then x is generated from squares.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[5] Wanda Szmielew. *From Affine to Euclidean Geometry.* Volume 27, PWN – D.Reidel Publ. Co., Warszawa – Dordrecht, 1983.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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