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Real Exponents and Logarithms [1]

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Summary. Definitions and properties of the following concepts: root, real exponent and logarithm. Also the number e is defined.

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The papers [11], [2], [9], [1], [7], [5], [6], [13], [12], [4], [3], [8], and [10] provide the notation and terminology for this paper. For simplicity we follow the rules: a, b, c, d denote real numbers, m, n, m_1, m_2 denote natural numbers, k, l denote integers, and p denotes a rational number. One can prove the following propositions:

(1) If there exists m such that $n = 2 \cdot m$, then $(-a)^n_{\mathbb{N}} = a^n_{\mathbb{N}}$.

(2) If there exists m such that $n = 2 \cdot m + 1$, then $(-a)^n_{\mathbb{N}} = -a^n_{\mathbb{N}}$.

(3) If $a \geq 0$ or there exists m such that $n = 2 \cdot m$, then $a^n_{\mathbb{N}} \geq 0$.

Let us consider n, a. The functor $\sqrt[n]{a}$ yields a real number and is defined by:

(Def.1) (i) $\sqrt[n]{a} = \mathrm{root}_n(a)$ if $a \geq 0$ and $n \geq 1$,

(ii) $\sqrt[n]{a} = -\mathrm{root}_n(-a)$ if $a < 0$ and there exists m such that $n = 2 \cdot m + 1$.

One can prove the following propositions:

(4) For all a, n holds if $a \geq 0$ and $n \geq 1$, then $\sqrt[n]{a} = \mathrm{root}_n(a)$ but if $a < 0$ and there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a} = -\mathrm{root}_n(-a)$.

(5) If $n \geq 1$ and $a \geq 0$ or there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a}^n_{\mathbb{N}} = a$ and $\sqrt[n]{a^n_{\mathbb{N}}} = a$.

(6) If $n \geq 1$, then $\sqrt[n]{0} = 0$.

(7) If $n \geq 1$, then $\sqrt[n]{1} = 1$.

(8) If $a \geq 0$ and $n \geq 1$, then $\sqrt[n]{a} \geq 0$.

(9) If there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{-1} = -1$.

(10) $\sqrt[1]{a} = a$.

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(11) If there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a} = -\sqrt[n]{-a}$.

(12) If $n \geq 1$ and $a \geq 0$ and $b \geq 0$ or there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

(13) If $a > 0$ and $n \geq 1$ or $a \neq 0$ and there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{\frac{1}{a}} = \frac{1}{\sqrt[n]{a}}$.

(14) If $a \geq 0$ and $b > 0$ and $n \geq 1$ or $b \neq 0$ and there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

(15) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$ or there exist m_1, m_2 such that $n = 2 \cdot m_1 + 1$ and $m = 2 \cdot m_2 + 1$, then $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$.

(16) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$ or there exist m_1, m_2 such that $n = 2 \cdot m_1 + 1$ and $m = 2 \cdot m_2 + 1$, then $\sqrt[n]{a} \cdot \sqrt[m]{a} = \sqrt[n \cdot m]{a_{\mathbb{N}}^{n+m}}$.

(17) If $a \leq b$ but $0 \leq a$ and $n \geq 1$ or there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

(18) If $a < b$ but $a \geq 0$ and $n \geq 1$ or there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

(19) If $a \geq 1$ and $n \geq 1$, then $\sqrt[n]{a} \geq 1$ and $a \geq \sqrt[n]{a}$.

(20) If $a \leq -1$ and there exists m such that $n = 2 \cdot m + 1$, then $\sqrt[n]{a} \leq -1$ and $a \leq \sqrt[n]{a}$.

(21) If $a \geq 0$ and $a < 1$ and $n \geq 1$, then $a \leq \sqrt[n]{a}$ and $\sqrt[n]{a} < 1$.

(22) If $a > -1$ and $a \leq 0$ and there exists m such that $n = 2 \cdot m + 1$, then $a \geq \sqrt[n]{a}$ and $\sqrt[n]{a} > -1$.

(23) If $a > 0$ and $n \geq 1$, then $\sqrt[n]{a} - 1 \leq \frac{a-1}{n}$.

(24) For every sequence of real numbers s and for every a such that $a > 0$ and for every n such that $n \geq 1$ holds $s(n) = \sqrt[n]{a}$ holds s is convergent and $\lim s = 1$.

Let us consider a, b. The functor a^b yielding a real number is defined as follows:

(Def.2) (i) $a^b = a_{\mathbb{R}}^b$ if $a > 0$,

(ii) $a^b = 0$ if $a = 0$ and $b > 0$,

(iii) there exists k such that $k = b$ and $a^b = a_{\mathbb{Z}}^k$ if $a < 0$ and b is an integer.

One can prove the following propositions:

(25) Given a, b. Then if $a > 0$, then $a^b = a_{\mathbb{R}}^b$ but if $a = 0$ and $b > 0$, then $a^b = 0$ but if $a < 0$ and b is an integer, then there exists k such that $k = b$ and $a^b = a_{\mathbb{Z}}^k$.

(26) If $a > 0$, then $a^b = a_{\mathbb{R}}^b$.

(27) If $b > 0$, then $0^b = 0$.

(28) If $a < 0$, then $a^k = a_{\mathbb{Z}}^k$.

(29) If $a \neq 0$, then $a^0 = 1$.

(30) $a^1 = a$.

(31) $1^a = 1$.

(32) If $a > 0$, then $a^{b+c} = a^b \cdot a^c$.

(33) If $a > 0$, then $a^{-c} = \frac{1}{a^c}$.

(34) If $a > 0$, then $a^{b-c} = \frac{a^b}{a^c}$.

(35) If $a > 0$ and $b > 0$, then $(a \cdot b)^c = a^c \cdot b^c$.

(36) If $a > 0$ and $b > 0$, then $\frac{a}{b}^c = \frac{a^c}{b^c}$.

(37) If $a > 0$, then $\frac{1}{a}^b = a^{-b}$.

(38) If $a > 0$, then $(a^b)^c = a^{b \cdot c}$.

(39) If $a > 0$, then $a^b > 0$.

(40) If $a > 1$ and $b > 0$, then $a^b > 1$.

(41) If $a > 1$ and $b < 0$, then $a^b < 1$.

(42) If $a > 0$ and $a < b$ and $c > 0$, then $a^c < b^c$.

(43) If $a > 0$ and $a < b$ and $c < 0$, then $a^c > b^c$.

(44) If $a < b$ and $c > 1$, then $c^a < c^b$.

(45) If $a < b$ and $c > 0$ and $c < 1$, then $c^a > c^b$.

(46) If $a \neq 0$, then $a^n = a^n_{\mathbb{N}}$.

(47) If $n \geq 1$, then $a^n = a^n_{\mathbb{N}}$.

(48) If $a \neq 0$, then $a^n = a^n$.

(49) If $n \geq 1$, then $a^n = a^n$.

(50) If $a \neq 0$, then $a^k = a^k_{\mathbb{Z}}$.

(51) If $a > 0$, then $a^p = a^p_{\mathbb{Q}}$.

(52) If $a \geq 0$ and $n \geq 1$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

(53) $a^2 = a^{\mathbf{2}}$.

(54) If $a \neq 0$ and there exists l such that $k = 2 \cdot l$, then $(-a)^k = a^k$.

(55) If $a \neq 0$ and there exists l such that $k = 2 \cdot l + 1$, then $(-a)^k = -a^k$.

Next we state two propositions:

(56) If $-1 < a$, then $(1 + a)^n \geq 1 + n \cdot a$.

(57) If $a > 0$ and $a \neq 1$ and $c \neq d$, then $a^c \neq a^d$.

Let us consider a, b. Let us assume that $a > 0$ and $a \neq 1$ and $b > 0$. The functor $\log_a b$ yields a real number and is defined by:

(Def.3) $a^{\log_a b} = b$.

The following propositions are true:

(58) For all a, b, c such that $a > 0$ and $a \neq 1$ and $b > 0$ holds $c = \log_a b$ if and only if $a^c = b$.

(59) If $a > 0$ and $a \neq 1$, then $\log_a 1 = 0$.

(60) If $a > 0$ and $a \neq 1$, then $\log_a a = 1$.

(61) If $a > 0$ and $a \neq 1$ and $b > 0$ and $c > 0$, then $\log_a b + \log_a c = \log_a (b \cdot c)$.

(62) If $a > 0$ and $a \neq 1$ and $b > 0$ and $c > 0$, then $\log_a b - \log_a c = \log_a \frac{b}{c}$.

(63) If $a > 0$ and $a \neq 1$ and $b > 0$, then $\log_a(b^c) = c \cdot \log_a b$.

(64) If $a > 0$ and $a \neq 1$ and $b > 0$ and $b \neq 1$ and $c > 0$, then $\log_a c = \log_a b \cdot \log_b c$.

(65) If $a > 1$ and $b > 0$ and $c > b$, then $\log_a c > \log_a b$.

(66) If $a > 0$ and $a < 1$ and $b > 0$ and $c > b$, then $\log_a c < \log_a b$.

(67) For every sequence of real numbers s such that for every n holds $s(n) = (1 + \frac{1}{n+1})^{n+1}$ holds s is convergent.

The real number e is defined as follows:

(Def.4) for every sequence of real numbers s such that for every n holds $s(n) = (1 + \frac{1}{n+1})^{n+1}$ holds $e = \lim s$.

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