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Incidence Projective Space (a reduction theorem in a plane) [1]

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Summary. The article begins with basic facts concernig arbitrary projective spaces. Further we are concerned with Fano projective spaces (we prove it has a rank of at least four). Finally we confine ourselves to Desarguesian planes; we define the notion of perspectivity and we prove the reduction theorem for projectivities with concurrent axes.

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The articles [6], [8], [5], [7], [9], [10], [4], [3], [1], and [2] provide the terminology and notation for this paper. We adopt the following convention: I_1 will be a projective space defined in terms of incidence, a, b, c, d, p, q, o, r, s will be elements of the points of I_1, and A, B, C, P, Q will be elements of the lines of I_1. We now state a number of propositions:

(1) There exists a such that $a \nmid A$.

(2) There exists A such that $a \nmid A$.

(3) If $A \neq B$, then there exist a, b such that $a \mid A$ and $a \nmid B$ and $b \mid B$ and $b \nmid A$.

(4) If $a \neq b$, then there exist A, B such that $a \mid A$ and $a \nmid B$ and $b \mid B$ and $b \nmid A$.

(5) There exist A, B, C such that $a \mid A$ and $a \mid B$ and $a \mid C$ and $A \neq B$ and $B \neq C$ and $C \neq A$.

(6) There exists a such that $a \nmid A$ and $a \nmid B$.

(7) There exists a such that $a \mid A$.

(8) If $a \mid A$ and $b \mid A$, then there exists c such that $c \mid A$ and $c \neq a$ and $c \neq b$.

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(9) There exists A such that $a \nmid A$ and $b \nmid A$.

(10) If $A \neq B$ and $o \mid A$ and $o \mid B$ and $p \mid A$ and $p \neq o$ and $q \mid B$, then $p \neq q$.

(11) If $o \neq a$ and $o \neq b$ and $A \neq B$ and $o \mid A$ and $o \mid B$ and $a \mid A$ and $a \mid C$ and $b \mid B$ and $b \mid C$, then $A \neq C$.

(12) Suppose $o \mid A$ and $o \mid B$ and $A \neq B$ and $a \mid A$ and $o \neq a$ and $b \mid B$ and $c \mid B$ and $b \neq c$ and $a \mid P$ and $b \mid P$ and $a \mid Q$ and $c \mid Q$. Then $P \neq Q$.

(13) If $a, b, c \mid A$, then $a, c, b \mid A$ and $b, a, c \mid A$ and $b, c, a \mid A$ and $c, a, b \mid A$ and $c, b, a \mid A$.

(14) Let I_1 be a Desarguesian projective space defined in terms of incidence. Let o, b_1, a_1, b_2, a_2, b_3, a_3, r, s, t be elements of the points of I_1. Let C_1, C_2, C_3, A_1, A_2, A_3, B_1, B_2, B_3 be elements of the lines of I_1. Suppose that

(i) $o, b_1, a_1 \mid C_1$,
(ii) $o, a_2, b_2 \mid C_2$,
(iii) $o, a_3, b_3 \mid C_3$,
(iv) $a_3, a_2, t \mid A_1$,
(v) $a_3, r, a_1 \mid A_2$,
(vi) $a_2, s, a_1 \mid A_3$,
(vii) $t, b_2, b_3 \mid B_1$,
(viii) $b_1, r, b_3 \mid B_2$,
(ix) $b_1, s, b_2 \mid B_3$,
(x) C_1, C_2, C_3 are mutually different,
(xi) $o \neq a_3$,
(xii) $o \neq b_1$,
(xiii) $o \neq b_2$,
(xiv) $a_2 \neq b_2$.

Then there exists an element O of the lines of I_1 such that $r, s, t \mid O$.

(15) Suppose there exist A, a, b, c, d such that $a \mid A$ and $b \mid A$ and $c \mid A$ and $d \mid A$ and a, b, c, d are mutually different. Then for every B there exist p, q, r, s such that $p \mid B$ and $q \mid B$ and $r \mid B$ and $s \mid B$ and p, q, r, s are mutually different.

We follow a convention: I_1 will be a Fanoian projective space defined in terms of incidence, a, b, c, d, p, q, r, s will be elements of the points of I_1, and A, B, C, D, L, Q, R, S will be elements of the lines of I_1. The following propositions are true:

(16) There exist p, q, r, s, a, b, c, A, B, C, Q, L, R, S, D such that $q \nmid L$ and $r \nmid L$ and $p \nmid Q$ and $s \nmid Q$ and $p \nmid R$ and $r \nmid R$ and $q \nmid S$ and $s \nmid S$ and $a, p, s \mid L$ and $a, q, r \mid Q$ and $b, q, s \mid R$ and $b, p, r \mid S$ and $c, p, q \mid A$ and $c, r, s \mid B$ and $a, b \mid C$ and $c \nmid C$.

(17) There exist a, A, B, C, D such that $a \mid A$ and $a \mid B$ and $a \mid C$ and $a \mid D$ and A, B, C, D are mutually different.

(18) There exist a, b, c, d, A such that $a \mid A$ and $b \mid A$ and $c \mid A$ and $d \mid A$ and a, b, c, d are mutually different.

(19) There exist p, q, r, s such that $p \mid B$ and $q \mid B$ and $r \mid B$ and $s \mid B$ and p, q, r, s are mutually different.

We follow a convention: I_1 will denote a Desarguesian 2-dimensional projective space defined in terms of incidence, c, p, q, x, y will denote elements of the points of I_1, and K, L, R, X will denote elements of the lines of I_1. Let us consider I_1, K, L, p. Let us assume that $p \nmid K$ and $p \nmid L$. The functor $\pi_p(K \to L)$ yields a partial function from the points of I_1 to the points of I_1 and is defined as follows:

(Def.1) $\operatorname{dom} \pi_p(K \to L) \subseteq$ the points of I_1 and for every x holds $x \in \operatorname{dom} \pi_p(K \to L)$ if and only if $x \mid K$ and for all x, y such that $x \mid K$ and $y \mid L$ holds $\pi_p(K \to L)(x) = y$ if and only if there exists X such that $p \mid X$ and $x \mid X$ and $y \mid X$.

One can prove the following propositions:

(20) Suppose $p \nmid K$ and $p \nmid L$. Then
 (i) $\operatorname{dom} \pi_p(K \to L) \subseteq$ the points of I_1,
 (ii) for every x holds $x \in \operatorname{dom} \pi_p(K \to L)$ if and only if $x \mid K$,
 (iii) for all x, y such that $x \mid K$ and $y \mid L$ holds $\pi_p(K \to L)(x) = y$ if and only if there exists X such that $p \mid X$ and $x \mid X$ and $y \mid X$.

(21) If $p \nmid K$, then for every x such that $x \mid K$ holds $\pi_p(K \to K)(x) = x$.

(22) If $p \nmid K$ and $p \nmid L$ and $x \mid K$, then $\pi_p(K \to L)(x)$ is an element of the points of I_1.

(23) If $p \nmid K$ and $p \nmid L$ and $x \mid K$ and $y = \pi_p(K \to L)(x)$, then $y \mid L$.

(24) If $p \nmid K$ and $p \nmid L$ and $y \in \operatorname{rng} \pi_p(K \to L)$, then $y \mid L$.

(25) Suppose $p \nmid K$ and $p \nmid L$ and $q \nmid L$ and $q \nmid R$. Then $\operatorname{dom}(\pi_q(L \to R) \cdot \pi_p(K \to L)) = \operatorname{dom} \pi_p(K \to L)$ and $\operatorname{rng}(\pi_q(L \to R) \cdot \pi_p(K \to L)) = \operatorname{rng} \pi_q(L \to R)$.

(26) Let a_1, b_1, a_2, b_2 be elements of the points of I_1. Then if $p \nmid K$ and $p \nmid L$ and $a_1 \mid K$ and $b_1 \mid K$ and $\pi_p(K \to L)(a_1) = a_2$ and $\pi_p(K \to L)(b_1) = b_2$ and $a_2 = b_2$, then $a_1 = b_1$.

(27) If $p \nmid K$ and $p \nmid L$ and $x \mid K$ and $x \mid L$, then $\pi_p(K \to L)(x) = x$.

We now state the proposition

(28) Suppose $p \nmid K$ and $p \nmid L$ and $q \nmid L$ and $q \nmid R$ and $c \mid K$ and $c \mid L$ and $c \mid R$ and $K \neq R$. Then there exists an element o of the points of I_1 such that $o \nmid K$ and $o \nmid R$ and $\pi_q(L \to R) \cdot \pi_p(K \to L) = \pi_o(K \to R)$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[2] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[3] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.
[4] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[5] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[7] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[9] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[10] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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