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Submetric Spaces - Part I [1]

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Summary. Definitions of pseudometric space, nonsymmetric metric space, semimetric space and ultrametric space are introduced. We find some relations between these spaces and prove that every ultrametric space is a metric space. We define the relation *is between*. Moreover we introduce the notions of the open segment and the closed segment.

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The terminology and notation used here are introduced in the following articles: [8], [2], [3], [1], [6], [4], [7], [9], and [5]. One can prove the following propositions:

(1) For all elements x, y of $\mathbb{R}$ such that $0 \leq x$ and $0 \leq y$ holds $\max(x, y) \leq x + y$.

(2) For every metric space M and for all elements x, y of the carrier of M such that $x \neq y$ holds $0 < \rho(x, y)$.

(3) For every element x of $\{\emptyset\}$ holds $x = \emptyset$.

(4) For all elements x, y of $\{\emptyset\}$ such that $x = y$ holds $\{[\emptyset, \emptyset]\} \mapsto 0(x,\, y) = 0$.

(5) For all elements x, y of $\{\emptyset\}$ such that $x \neq y$ holds $0 < \{[\emptyset, \emptyset]\} \mapsto 0(x,\, y)$.

(6) For all elements x, y of $\{\emptyset\}$ holds $\{[\emptyset, \emptyset]\} \mapsto 0(x,\, y) = \{[\emptyset, \emptyset]\} \mapsto 0(y,\, x)$.

(7) For all elements x, y, z of $\{\emptyset\}$ holds $\{[\emptyset, \emptyset]\} \mapsto 0(x,\, z) \leq \{[\emptyset, \emptyset]\} \mapsto 0(x,\, y) + \{[\emptyset, \emptyset]\} \mapsto 0(y,\, z)$.

(8) For all elements x, y, z of $\{\emptyset\}$ holds $\{[\emptyset, \emptyset]\} \mapsto 0(x,\, z) \leq \max(\{[\emptyset, \emptyset]\} \mapsto 0(x,\, y), \{[\emptyset, \emptyset]\} \mapsto 0(y,\, z))$.

A metric structure is called a pseudo metric space if:

(Def.1) for all elements a, b, c of the carrier of it holds if $a = b$, then $\rho(a, b) = 0$ but $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

Next we state four propositions:

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(10)[2] For every pseudo metric space M and for all elements a, b of the carrier of M such that $a = b$ holds $\rho(a, b) = 0$.

(11) For every pseudo metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = \rho(b, a)$.

(12) For every pseudo metric space M and for all elements a, b, c of the carrier of M holds $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

(13) For every pseudo metric space M and for all elements a, b of the carrier of M holds $0 \leq \rho(a, b)$.

A metric structure is said to be a semi metric space if:

(Def.2) for all elements a, b of the carrier of it holds if $a = b$, then $\rho(a, b) = 0$ but if $a \neq b$, then $0 < \rho(a, b)$ and $\rho(a, b) = \rho(b, a)$.

One can prove the following four propositions:

(15)[3] For every semi metric space M and for all elements a, b of the carrier of M such that $a = b$ holds $\rho(a, b) = 0$.

(16) For every semi metric space M and for all elements a, b of the carrier of M such that $a \neq b$ holds $0 < \rho(a, b)$.

(17) For every semi metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = \rho(b, a)$.

(18) For every semi metric space M and for all elements a, b of the carrier of M holds $0 \leq \rho(a, b)$.

A metric structure is called a non-symmetric metric space if:

(Def.3) for all elements a, b, c of the carrier of it holds if $a = b$, then $\rho(a, b) = 0$ but if $a \neq b$, then $0 < \rho(a, b)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

One can prove the following four propositions:

(20)[4] For every non-symmetric metric space M and for all elements a, b of the carrier of M such that $a = b$ holds $\rho(a, b) = 0$.

(21) For every non-symmetric metric space M and for all elements a, b of the carrier of M such that $a \neq b$ holds $0 < \rho(a, b)$.

(22) For every non-symmetric metric space M and for all elements a, b, c of the carrier of M holds $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

(23) For every non-symmetric metric space M and for all elements a, b of the carrier of M holds $0 \leq \rho(a, b)$.

A metric structure is said to be a ultra metric space if:

(Def.4) for all elements a, b, c of the carrier of it holds if $a = b$, then $\rho(a, b) = 0$ but if $a \neq b$, then $0 < \rho(a, b)$ and $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \max(\rho(a, b), \rho(b, c))$.

We now state a number of propositions:

[2]The proposition (9) was either repeated or obvious.
[3]The proposition (14) was either repeated or obvious.
[4]The proposition (19) was either repeated or obvious.

(25)[5] For every ultra metric space M and for all elements a, b of the carrier of M such that $a = b$ holds $\rho(a, b) = 0$.

(26) For every ultra metric space M and for all elements a, b of the carrier of M such that $a \neq b$ holds $0 < \rho(a, b)$.

(27) For every ultra metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = \rho(b, a)$.

(28) For every ultra metric space M and for all elements a, b, c of the carrier of M holds $\rho(a, c) \leq \max(\rho(a, b), \rho(b, c))$.

(29) For every ultra metric space M and for all elements a, b of the carrier of M holds $0 \leq \rho(a, b)$.

(30) For every metric space M holds M is a pseudo metric space.

(31) For every metric space M holds M is a semi metric space.

(32) For every metric space M holds M is a non-symmetric metric space.

(33) For every ultra metric space M holds M is a metric space.

(34) For every ultra metric space M holds M is a pseudo metric space.

(35) For every ultra metric space M holds M is a semi metric space.

(36) For every ultra metric space M holds M is a non-symmetric metric space.

In the sequel x, y will be arbitrary. Let us consider x, y. Then $\{x, y\}$ is a non-empty set.

The function $(2^2 \to 0)$ from $[\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!]$ into $\mathbb{R}$ is defined by:

(Def.5) $(2^2 \to 0) = [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!] \longmapsto 0$.

Next we state several propositions:

(37) $(2^2 \to 0) = [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!] \longmapsto 0$.

(38) For every element x of $\{\emptyset, \{\emptyset\}\}$ holds $x = \emptyset$ or $x = \{\emptyset\}$.

(39) (i) $\langle \emptyset, \emptyset \rangle \in [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!]$,
(ii) $\langle \emptyset, \{\emptyset\} \rangle \in [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!]$,
(iii) $\langle \{\emptyset\}, \emptyset \rangle \in [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!]$,
(iv) $\langle \{\emptyset\}, \{\emptyset\} \rangle \in [\!:\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}:\!]$.

(40) For all elements x, y of $\{\emptyset, \{\emptyset\}\}$ holds $(2^2 \to 0)(x, y) = 0$.

(41) For all elements x, y of $\{\emptyset, \{\emptyset\}\}$ such that $x = y$ holds $(2^2 \to 0)(x, y) = 0$.

(42) For all elements x, y of $\{\emptyset, \{\emptyset\}\}$ holds $(2^2 \to 0)(x, y) = (2^2 \to 0)(y, x)$.

(43) For all elements x, y, z of $\{\emptyset, \{\emptyset\}\}$ holds $(2^2 \to 0)(x, z) \leq (2^2 \to 0)(x, y) + (2^2 \to 0)(y, z)$.

The pseudo metric space $\ominus$ is defined as follows:

(Def.6) $\ominus = \langle \{\emptyset, \{\emptyset\}\}, (2^2 \to 0) \rangle$.

The following proposition is true

(44) $\ominus = \langle \{\emptyset, \{\emptyset\}\}, (2^2 \to 0) \rangle$.

[5]The proposition (24) was either repeated or obvious.

Let S be a metric space, and let p, q, r be elements of the carrier of S. We say that q is between p and r if and only if:

(Def.7) $p \neq q$ and $p \neq r$ and $q \neq r$ and $\rho(p,r) = \rho(p,q) + \rho(q,r)$.

Next we state three propositions:

(47)[6] For every metric space S and for all elements p, q, r of the carrier of S such that q is between p and r holds q is between r and p.

(48) For every metric space S and for all elements p, q, r of the carrier of S such that q is between p and r holds p is not between q and r and r is not between p and q.

(49) For every metric space S and for all elements p, q, r, s of the carrier of S such that q is between p and r and r is between p and s holds q is between p and s and r is between q and s.

Let M be a metric space, and let p, r be elements of the carrier of M. The functor $\mathrm{IntSeg}(p,r)$ yielding a subset of the carrier of M is defined as follows:

(Def.8) $\mathrm{IntSeg}(p,r) = \{q : q$ is between p and r $\}$, where q ranges over elements of the carrier of M.

One can prove the following two propositions:

(50) For every metric space M and for all elements p, r of the carrier of M holds $\mathrm{IntSeg}(p,r) = \{q : q$ is between p and r $\}$, where q ranges over elements of the carrier of M.

(51) For every metric space M and for all elements p, r, x of the carrier of M holds $x \in \mathrm{IntSeg}(p,r)$ if and only if x is between p and r.

Let M be a metric space, and let p, r be elements of the carrier of M. The functor $\mathrm{ClSeg}(p,r)$ yielding a subset of the carrier of M is defined by:

(Def.9) $\mathrm{ClSeg}(p,r) = \{q : q$ is between p and r $\} \cup \{p,r\}$, where q ranges over elements of the carrier of M.

We now state three propositions:

(52) For every metric space M and for all elements p, r of the carrier of M holds $\mathrm{ClSeg}(p,r) = \{q : q$ is between p and r $\} \cup \{p,r\}$, where q ranges over elements of the carrier of M.

(53) For every metric space M and for all elements p, r, x of the carrier of M holds $x \in \mathrm{ClSeg}(p,r)$ if and only if x is between p and r or $x = p$ or $x = r$.

(54) For every metric space M and for all elements p, r of the carrier of M holds $\mathrm{IntSeg}(p,r) \subseteq \mathrm{ClSeg}(p,r)$.

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