FORMALIZED MATHEMATICS
Vol.2, No.**3**, May–August 1991
Université Catholique de Louvain

Fundamental Types of Metric Affine Spaces

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We distinguish in the class of metric affine spaces some fundamental types of them. First we can assume the underlying affine space to satisfy classical affine configurational axiom; thus we come to Pappian, Desarguesian, Moufangian, and translation spaces. Next we distinguish the spaces satisfying theorem on three perpendiculars and the homogeneous spaces; these properties directly refer to some axioms involving orthogonality. Some known relationships between the introduced classes of structures are established. We also show that the commonly investigated models of metric affine geometry constructed in a real linear space with the help of a symmetric bilinear form belong to all the classes introduced in the paper.

MML Identifier: `EUCLMETR`.

The papers [1], [3], [5], [6], [2], [4], [7], [8], and [9] provide the notation and terminology for this paper. A metric affine space is Euclidean if:

(Def.1) for all elements a, b, c, d of the points of it such that $a,b \perp c,d$ and $b,c \perp a,d$ holds $b,d \perp a,c$.

A metric affine space is Pappian if:

(Def.2) the affine reduct of it is Pappian.

A metric affine space is Desarguesian if:

(Def.3) the affine reduct of it is Desarguesian.

A metric affine space is Fanoian if:

(Def.4) the affine reduct of it is Fanoian.

A metric affine space is Moufangian if:

(Def.5) the affine reduct of it is Moufangian.

A metric affine space is translation if:

(Def.6) the affine reduct of it is translation.

A metric affine space is homogeneous if it satisfies the condition (Def.7).

ISSN 0777–4028

(Def.7) Let o, a, a_1, b, b_1, c, c_1 be elements of the points of it . Then if $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$, then $b, c \perp b_1, c_1$.

In the sequel M_1 denotes a metric affine plane and M_2 denotes a metric affine space. The following propositions are true:

(1) For all elements a, b, c of the points of M_2 such that not $\mathbf{L}(a, b, c)$ holds $a \neq b$ and $b \neq c$ and $a \neq c$.

(2) For all elements a, b, c, d of the points of M_1 and for every subset K of the points of M_1 such that $a, b \perp K$ and $c, d \perp K$ holds $a, b \parallel c, d$ and $a, b \parallel d, c$.

(3) For all elements a, b of the points of M_1 and for all subsets A, K of the points of M_1 such that $a \neq b$ but $a, b \perp K$ or $b, a \perp K$ but $a, b \perp A$ or $b, a \perp A$ holds $K \parallel A$.

(4) For all elements x, y, z of the points of M_2 such that $\mathbf{L}(x, y, z)$ holds $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(5) For all elements a, b, c of the points of M_1 such that not $\mathbf{L}(a, b, c)$ there exists an element d of the points of M_1 such that $d, a \perp b, c$ and $d, b \perp a, c$.

(6) For all elements a, b, c, d_1, d_2 of the points of M_1 such that not $\mathbf{L}(a, b, c)$ and $d_1, a \perp b, c$ and $d_1, b \perp a, c$ and $d_2, a \perp b, c$ and $d_2, b \perp a, c$ holds $d_1 = d_2$.

(7) For all elements a, b, c, d of the points of M_1 such that $a, b \perp c, d$ and $b, c \perp a, d$ and $\mathbf{L}(a, b, c)$ holds $a = c$ or $a = b$ or $b = c$.

(8) M_1 is Euclidean if and only if theorem on three perpendiculars holds in M_1.

(9) M_1 is homogeneous if and only if othogonal verion of Desargues Axiom holds in M_1.

(10) M_1 is Pappian if and only if Pappos Axiom holds in M_1.

(11) M_1 is Desarguesian if and only if Desargues Axiom holds in M_1.

(12) M_1 is Moufangian if and only if trapezium variant of Desargues Axiom holds in M_1.

(13) M_1 is translation if and only if minor Desargues Axiom holds in M_1.

(14) If M_1 is homogeneous, then M_1 is Desarguesian.

(15) If M_1 is Euclidean Desarguesian, then M_1 is Pappian.

We adopt the following rules: V will denote a real linear space and w, y, u, v will denote vectors of V. The following propositions are true:

(16) Let o, c, c_1, a, a_1, a_2 be elements of the points of M_1. Then if not $\mathbf{L}(o, c, a)$ and $o \neq c_1$ and $o, c \perp o, c_1$ and $o, a \perp o, a_1$ and $o, a \perp o, a_2$ and $c, a \perp c_1, a_1$ and $c, a \perp c_1, a_2$, then $a_1 = a_2$.

(17) For all elements o, c, c_1, a of the points of M_1 such that not $\mathbf{L}(o, c, a)$ and $o \neq c_1$ and $o, c \perp o, c_1$ there exists an element a_1 of the points of M_1 such that $o, a \perp o, a_1$ and $c, a \perp c_1, a_1$.

(18) Let a, b be real numbers. Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y and $u = a \cdot w + b \cdot y$. Then there exists a real number c such that $c \neq 0$ and $v = c \cdot b \cdot w + (-c \cdot a) \cdot y$.

(19) Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y. Then there exists a real number c such that for all real numbers a, b holds $a \cdot w + b \cdot y$, $c \cdot b \cdot w + (-c \cdot a) \cdot y$ are orthogonal w.r.t. w, y and $(a \cdot w + b \cdot y) - u$, $(c \cdot b \cdot w + (-c \cdot a) \cdot y) - v$ are orthogonal w.r.t. w, y.

(20) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then for an arbitrary x holds x is a vector of V if and only if x is an element of the points of M_1.

(21) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then LIN holds in M_1.

(22) Suppose w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$. Let o, a, a_1, b, b_1, c, c_1 be elements of the points of M_1. Suppose $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$ and $o = a_1$. Then $b, c \perp b_1, c_1$.

(23) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then M_1 is homogeneous.

The following proposition is true

(24) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then M_1 is a metric affine plane.

Let M_1 be an Pappian metric affine plane. Then the affine reduct of M_1 is a Pappian affine plane.

Let M_1 be a Desarguesian metric affine plane. Then the affine reduct of M_1 is a Desarguesian affine plane.

Let M_1 be a Moufangian metric affine plane. Then the affine reduct of M_1 is a Moufangian affine plane.

Let M_1 be a translation metric affine plane. Then the affine reduct of M_1 is an translation affine plane.

Let M_1 be an Fanoian metric affine plane. Then the affine reduct of M_1 is a Fanoian affine plane.

Let M_1 be a homogeneous metric affine plane. Then the affine reduct of M_1 is an Desarguesian affine plane.

Let M_1 be a Euclidean Desarguesian metric affine plane. Then
the affine reduct of M_1
is a Pappian affine plane.

Let M_1 be an Pappian metric affine space. Then the affine reduct of M_1 is a Pappian affine space.

Let M_1 be a Desarguesian metric affine space. Then the affine reduct of M_1 is a Desarguesian affine space.

Let M_1 be an Moufangian metric affine space. Then the affine reduct of M_1 is an Moufangian affine space.

Let M_1 be a translation metric affine space. Then the affine reduct of M_1 is a translation affine space.

Let M_1 be a Fanoian metric affine space. Then the affine reduct of M_1 is a Fanoian affine space.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical metric affine spaces and planes. *Formalized Mathematics*, 1(**5**):891–899, 1990.
[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.
[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. A construction of analytical ordered trapezium spaces. *Formalized Mathematics*, 2(**3**):315–322, 1991.
[5] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity - part I. *Formalized Mathematics*, 1(**3**):611–615, 1990.
[6] Henryk Oryszczyszyn and Krzysztof Prażmowski. Parallelity and lines in affine spaces. *Formalized Mathematics*, 1(**3**):617–621, 1990.
[7] Krzysztof Prażmowski. Fanoian, Pappian and Desarguesian affine spaces. *Formalized Mathematics*, 2(**3**):341–346, 1991.
[8] Jolanta Świerzyńska and Bogdan Świerzyński. Metric-affine configurations in metric affine planes - Part II. *Formalized Mathematics*, 2(**3**):335–340, 1991.
[9] Jolanta Świerzyńska and Bogdan Świerzyński. Metric-affine configurations in metric affine planes - Part I. *Formalized Mathematics*, 2(**3**):331–334, 1991.

Received April 17, 1991