FORMALIZED MATHEMATICS
Vol.2, No.3, May–August 1991
Université Catholique de Louvain

# Monotonic and Continuous Real Function

Jarosław Kotowicz
Warsaw University
Białystok

**Summary.** A continuation of [16] and [13]. We prove a few theorems about real functions monotonic and continuous on interval, on halfline and on the set of real numbers and continuity of the inverse function. At the begining of the paper we show some facts about topological properties of the set of real numbers, halflines and intervals which rather belong to [17]

MML Identifier: `FCONT_3`.

The notation and terminology used in this paper are introduced in the following articles: [18], [5], [1], [2], [3], [20], [12], [6], [8], [15], [14], [4], [19], [9], [10], [17], [11], [16], and [7]. For simplicity we follow the rules: $X$ will denote a set, $x_0$, $r$, $r_1$, $g$, $p$ will denote real numbers, $n$ will denote a natural number, $a$ will denote a sequence of real numbers, and $f$ will denote a partial function from $\mathbb{R}$ to $\mathbb{R}$. Next we state several propositions:

(1) $\Omega_{\mathbb{R}}$ is closed.

(2) $\emptyset_{\mathbb{R}}$ is open.

(3) $\emptyset_{\mathbb{R}}$ is closed.

(4) $\Omega_{\mathbb{R}}$ is open.

(5) $[r, +\infty[$ is closed.

(6) $]-\infty, r]$ is closed.

(7) $]r, +\infty[$ is open.

(8) $]-\infty, r[$ is open.

Let us consider $r$. Then $]r, +\infty[$ is a real open subset. Then $\mathrm{HL}(r)$ is a real open subset.

Let us consider $p$, $g$. Then $]p, g[$ is a real open subset.

Next we state a number of propositions:

(9) $0 < r$ and $g \in \, ]x_0 - r, x_0 + r[$ if and only if there exists $r_1$ such that $g = x_0 + r_1$ and $|r_1| < r$.



© 1991 Fondation Philippe le Hodey
ISSN 0777–4028

(10) $0 < r$ and $g \in ]x_0 - r, x_0 + r[$ if and only if $g - x_0 \in ]-r, r[$.

(11) $]-\infty, p] = \{p\} \cup ]-\infty, p[$.

(12) $[p, +\infty[ = \{p\} \cup ]p, +\infty[$.

(13) If for every $n$ holds $a(n) = x_0 - \frac{p}{n+1}$, then $a$ is convergent and $\lim a = x_0$.

(14) If for every $n$ holds $a(n) = x_0 + \frac{p}{n+1}$, then $a$ is convergent and $\lim a = x_0$.

(15) If $f$ is continuous in $x_0$ and $f(x_0) \neq r$ and there exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \operatorname{dom} f$, then there exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \operatorname{dom} f$ and for every $g$ such that $g \in N$ holds $f(g) \neq r$.

(16) If $f$ is increasing on $X$ or $f$ is decreasing on $X$, then $f \restriction X$ is one-to-one.

(17) If $f$ is increasing on $X$, then $(f \restriction X)^{-1}$ is increasing on $f^{\circ} X$.

(18) If $f$ is decreasing on $X$, then $(f \restriction X)^{-1}$ is decreasing on $f^{\circ} X$.

(19) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and there exists $p$ such that $f^{\circ} X = ]-\infty, p[$, then $f$ is continuous on $X$.

(20) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and there exists $p$ such that $f^{\circ} X = ]p, +\infty[$, then $f$ is continuous on $X$.

(21) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and there exists $p$ such that $f^{\circ} X = ]-\infty, p]$, then $f$ is continuous on $X$.

(22) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and there exists $p$ such that $f^{\circ} X = [p, +\infty[$, then $f$ is continuous on $X$.

(23) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and there exist $p$, $g$ such that $f^{\circ} X = ]p, g[$, then $f$ is continuous on $X$.

(24) If $X \subseteq \operatorname{dom} f$ and $f$ is monotone on $X$ and $f^{\circ} X = \mathbb{R}$, then $f$ is continuous on $X$.

(25) If $f$ is increasing on $]p, g[$ or $f$ is decreasing on $]p, g[$ but $]p, g[ \subseteq \operatorname{dom} f$, then $(f \restriction ]p, g[)^{-1}$ is continuous on $f^{\circ} ]p, g[$.

(26) If $f$ is increasing on $]-\infty, p[$ or $f$ is decreasing on $]-\infty, p[$ but $]-\infty, p[ \subseteq \operatorname{dom} f$, then $(f \restriction ]-\infty, p[)^{-1}$ is continuous on $f^{\circ} ]-\infty, p[$.

(27) If $f$ is increasing on $]p, +\infty[$ or $f$ is decreasing on $]p, +\infty[$ but $]p, +\infty[ \subseteq \operatorname{dom} f$, then $(f \restriction ]p, +\infty[)^{-1}$ is continuous on $f^{\circ} ]p, +\infty[$.

(28) If $f$ is increasing on $]-\infty, p]$ or $f$ is decreasing on $]-\infty, p]$ but $]-\infty, p] \subseteq \operatorname{dom} f$, then $(f \restriction ]-\infty, p])^{-1}$ is continuous on $f^{\circ} ]-\infty, p]$.

(29) If $f$ is increasing on $[p, +\infty[$ or $f$ is decreasing on $[p, +\infty[$ but $[p, +\infty[ \subseteq \operatorname{dom} f$, then $(f \restriction [p, +\infty[)^{-1}$ is continuous on $f^{\circ} [p, +\infty[$.

(30) If $f$ is increasing on $\Omega_{\mathbb{R}}$ or $f$ is decreasing on $\Omega_{\mathbb{R}}$ but $f$ is total, then $f^{-1}$ is continuous on $\operatorname{rng} f$.

(31) If $f$ is continuous on $]p, g[$ but $f$ is increasing on $]p, g[$ or $f$ is decreasing on $]p, g[$, then $\operatorname{rng}(f \restriction ]p, g[)$ is open.

(32) If $f$ is continuous on $]-\infty, p[$ but $f$ is increasing on $]-\infty, p[$ or $f$ is decreasing on $]-\infty, p[$, then $\operatorname{rng}(f \restriction ]-\infty, p[)$ is open.

(33) If $f$ is continuous on $]p, +\infty[$ but $f$ is increasing on $]p, +\infty[$ or $f$ is decreasing on $]p, +\infty[$, then $\operatorname{rng}(f \restriction ]p, +\infty[)$ is open.

(34) If $f$ is continuous on $\Omega_{\mathbb{R}}$ but $f$ is increasing on $\Omega_{\mathbb{R}}$ or $f$ is decreasing on $\Omega_{\mathbb{R}}$, then $\mathrm{rng}\, f$ is open.

## References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[7] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.
[8] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[9] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[10] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[11] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[13] Jarosław Kotowicz and Konrad Raczkowski. Real function uniform continuity. *Formalized Mathematics*, 1(**4**):793–795, 1990.
[14] Andrzej Nędzusiak. $\sigma$-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[15] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[16] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[17] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

*Received January 10, 1991*