FORMALIZED MATHEMATICS
Vol.2, No.3, May–August 1991
Université Catholique de Louvain

Real Function Differentiability - Part II

Jarosław Kotowicz
Warsaw University
Białystok

Konrad Raczkowski
Warsaw University
Białystok

Summary. A continuation of [18]. We prove an equivalent definition of the derivative of the real function at the point and theorems about derivative of composite functions, inverse function and derivative of quotient of two functions. At the begining of the paper a few facts which rather belong to [8], [10], [7] are proved.

MML Identifier: FDIFF_2.

The terminology and notation used in this paper have been introduced in the following papers: [20], [5], [1], [2], [3], [22], [14], [8], [10], [16], [15], [4], [21], [11], [12], [19], [13], [17], [18], [9], and [6]. For simplicity we adopt the following convention: x_0, r, r_1, r_2, g, p will be real numbers, n, m will be natural numbers, a, b, d will be sequences of real numbers, h, h_1, h_2 will be real sequences convergent to 0, c will be a constant real sequence, A will be a real open subset, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider h. Then $-h$ is a real sequence convergent to 0.

The following propositions are true:

(1) If a is convergent and b is convergent and $\lim a = \lim b$ and for every n holds $d(2 \cdot n) = a(n)$ and $d(2 \cdot n + 1) = b(n)$, then d is convergent and $\lim d = \lim a$.

(2) If for every n holds $a(n) = 2 \cdot n$, then a is an increasing sequence of naturals.

(3) If for every n holds $a(n) = 2 \cdot n + 1$, then a is an increasing sequence of naturals.

(4) If $\operatorname{rng} c = \{x_0\}$, then c is convergent and $\lim c = x_0$ and $h + c$ is convergent and $\lim(h + c) = x_0$.

(5) If $\operatorname{rng} a = \{r\}$ and $\operatorname{rng} b = \{r\}$, then $a = b$.

(6) If a is a subsequence of h, then a is a real sequence convergent to 0.

ISSN 0777–4028

(7) Suppose for all h, c such that $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent. Given h_1, h_2, c. Suppose $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h_1+c) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(h_2+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$. Then $\lim({h_1}^{-1}(f \cdot (h_1+c) - f \cdot c)) = \lim({h_2}^{-1}(f \cdot (h_2+c) - f \cdot c))$.

(8) If there exists a neighbourhood N of r such that $N \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{r\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{r\} \subseteq \operatorname{dom} f$.

(9) If $\operatorname{rng} a \subseteq \operatorname{dom}(f_2 \cdot f_1)$, then $\operatorname{rng} a \subseteq \operatorname{dom} f_1$ and $\operatorname{rng}(f_1 \cdot a) \subseteq \operatorname{dom} f_2$.

The scheme *ExInc_Seq_of_Nat* concerns a sequence of real numbers $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an increasing sequence q of naturals such that for every n holds $\mathcal{P}[(\mathcal{A} \cdot q)(n)]$ and for every n such that for every r such that $r = \mathcal{A}(n)$ holds $\mathcal{P}[r]$ there exists m such that $n = q(m)$

provided the following requirement is met:

- for every n there exists m such that $n \leq m$ and $\mathcal{P}[\mathcal{A}(m)]$.

One can prove the following propositions:

(10) If $f(x_0) \neq r$ and f is differentiable in x_0, then there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for every g such that $g \in N$ holds $f(g) \neq r$.

(11) f is differentiable in x_0 if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent.

(12) f is differentiable in x_0 and $f'(x_0) = g$ if and only if the following conditions are satisfied:

(i) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent and $\lim(h^{-1}(f \cdot (h+c) - f \cdot c)) = g$.

(13) If f_1 is differentiable in x_0 and f_2 is differentiable in $f_1(x_0)$, then $f_2 \cdot f_1$ is differentiable in x_0 and $(f_2 \cdot f_1)'(x_0) = {f_2}'(f_1(x_0)) \cdot {f_1}'(x_0)$.

(14) If $f_2(x_0) \neq 0$ and f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $\frac{f_1}{f_2}$ is differentiable in x_0 and $(\frac{f_1}{f_2})'(x_0) = \frac{{f_1}'(x_0) \cdot f_2(x_0) - {f_2}'(x_0) \cdot f_1(x_0)}{f_2(x_0)^{\mathbf{2}}}$.

(15) If $f(x_0) \neq 0$ and f is differentiable in x_0, then $\frac{1}{f}$ is differentiable in x_0 and $(\frac{1}{f})'(x_0) = -\frac{f'(x_0)}{f(x_0)^{\mathbf{2}}}$.

(16) If f is differentiable on A, then $f \restriction A$ is differentiable on A and $f'_{\restriction A} = (f \restriction A)'_{\restriction A}$.

(17) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 + f_2$ is differentiable on A and $(f_1 + f_2)'_{\restriction A} = {f_1}'_{\restriction A} + {f_2}'_{\restriction A}$.

(18) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 - f_2$ is differentiable on A and $(f_1 - f_2)'_{\restriction A} = {f_1}'_{\restriction A} - {f_2}'_{\restriction A}$.

(19) If f is differentiable on A, then rf is differentiable on A and $(rf)'_{\restriction A} = rf'_{\restriction A}$.

(20) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 f_2$ is differentiable on A and $(f_1 f_2)'_{\restriction A} = {f_1}'_{\restriction A} f_2 + f_1 {f_2}'_{\restriction A}$.

(21) If f_1 is differentiable on A and f_2 is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f_2(x_0) \neq 0$, then $\frac{f_1}{f_2}$ is differentiable on A and $(\frac{f_1}{f_2})'_{\restriction A} = \frac{{f_1}'_{\restriction A} f_2 - {f_2}'_{\restriction A} f_1}{f_2 f_2}$.

(22) If f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f(x_0) \neq 0$, then $\frac{1}{f}$ is differentiable on A and $(\frac{1}{f})'_{\restriction A} = -\frac{f'_{\restriction A}}{f f}$.

(23) If f_1 is differentiable on A and $f_1 \circ A$ is a real open subset and f_2 is differentiable on $f_1 \circ A$, then $f_2 \cdot f_1$ is differentiable on A and $(f_2 \cdot f_1)'_{\restriction A} = ({f_2}'_{\restriction f_1 \circ A} \cdot f_1) {f_1}'_{\restriction A}$.

(24) If $A \subseteq \operatorname{dom} f$ and for all r, p such that $r \in A$ and $p \in A$ holds $|f(r) - f(p)| \leq (r - p)^{\mathbf{2}}$, then f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f'(x_0) = 0$.

(25) Suppose for all r_1, r_2 such that $r_1 \in]p, g[$ and $r_2 \in]p, g[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$ and $p < g$ and $]p, g[\subseteq \operatorname{dom} f$. Then f is differentiable on $]p, g[$ and f is a constant on $]p, g[$.

(26) If $]-\infty, r[\subseteq \operatorname{dom} f$ and for all r_1, r_2 such that $r_1 \in]-\infty, r[$ and $r_2 \in]-\infty, r[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $]-\infty, r[$ and f is a constant on $]-\infty, r[$.

(27) If $]r, +\infty[\subseteq \operatorname{dom} f$ and for all r_1, r_2 such that $r_1 \in]r, +\infty[$ and $r_2 \in]r, +\infty[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $]r, +\infty[$ and f is a constant on $]r, +\infty[$.

(28) If f is total and for all r_1, r_2 holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $\Omega_{\mathbb{R}}$ and f is a constant on $\Omega_{\mathbb{R}}$.

(29) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $0 < f'(x_0)$, then f is increasing on $]-\infty, r[$ and $f \restriction]-\infty, r[$ is one-to-one.

(30) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $f'(x_0) < 0$, then f is decreasing on $]-\infty, r[$ and $f \restriction]-\infty, r[$ is one-to-one.

(31) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $0 \leq f'(x_0)$, then f is non-decreasing on $]-\infty, r[$.

(32) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $f'(x_0) \leq 0$, then f is non-increasing on $]-\infty, r[$.

(33) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$ holds $0 < f'(x_0)$, then f is increasing on $]r, +\infty[$ and $f \restriction]r, +\infty[$ is one-to-one.

(34) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$ holds $f'(x_0) < 0$, then f is decreasing on $]r, +\infty[$ and $f \restriction]r, +\infty[$ is one-to-one.

(35) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$

holds $0 \leq f'(x_0)$, then f is non-decreasing on $]r, +\infty[$.

(36) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in\,]r, +\infty[$ holds $f'(x_0) \leq 0$, then f is non-increasing on $]r, +\infty[$.

(37) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 < f'(x_0)$, then f is increasing on $\Omega_{\mathbb{R}}$ and f is one-to-one.

(38) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) < 0$, then f is decreasing on $\Omega_{\mathbb{R}}$ and f is one-to-one.

(39) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 \leq f'(x_0)$, then f is non-decreasing on $\Omega_{\mathbb{R}}$.

(40) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) \leq 0$, then f is non-increasing on $\Omega_{\mathbb{R}}$.

One can prove the following propositions:

(41) If f is differentiable on $]p, g[$ but for every x_0 such that $x_0 \in\,]p, g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]p, g[$ holds $f'(x_0) < 0$, then $\mathrm{rng}(f \restriction]p, g[)$ is open.

(42) If f is differentiable on $]-\infty, p[$ but for every x_0 such that $x_0 \in\,]-\infty, p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]-\infty, p[$ holds $f'(x_0) < 0$, then $\mathrm{rng}(f \restriction]-\infty, p[)$ is open.

(43) If f is differentiable on $]p, +\infty[$ but for every x_0 such that $x_0 \in\,]p, +\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]p, +\infty[$ holds $f'(x_0) < 0$, then $\mathrm{rng}(f \restriction]p, +\infty[)$ is open.

(44) If f is differentiable on $\Omega_{\mathbb{R}}$ but for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$, then $\mathrm{rng}\, f$ is open.

(45) Suppose f is differentiable on $\Omega_{\mathbb{R}}$ but for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$. Then f is one-to-one and f^{-1} is differentiable on $\mathrm{dom}(f^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}(f^{-1})$ holds $(f^{-1})'(x_0) = \frac{1}{f'(f^{-1}(x_0))}$.

(46) Suppose f is differentiable on $]-\infty, p[$ but for every x_0 such that $x_0 \in\,]-\infty, p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]-\infty, p[$ holds $f'(x_0) < 0$. Then $f \restriction]-\infty, p[$ is one-to-one and $(f \restriction]-\infty, p[)^{-1}$ is differentiable on $\mathrm{dom}((f \restriction]-\infty, p[)^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}((f \restriction]-\infty, p[)^{-1})$ holds $((f \restriction]-\infty, p[)^{-1})'(x_0) = \frac{1}{f'((f \restriction]-\infty, p[)^{-1}(x_0))}$.

(47) Suppose f is differentiable on $]p, +\infty[$ but for every x_0 such that $x_0 \in\,]p, +\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]p, +\infty[$ holds $f'(x_0) < 0$. Then $f \restriction]p, +\infty[$ is one-to-one and $(f \restriction]p, +\infty[)^{-1}$ is differentiable on $\mathrm{dom}((f \restriction]p, +\infty[)^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}((f \restriction]p, +\infty[)^{-1})$ holds $((f \restriction]p, +\infty[)^{-1})'(x_0) = \frac{1}{f'((f \restriction]p, +\infty[)^{-1}(x_0))}$.

(48) Suppose f is differentiable on $]p, g[$ but for every x_0 such that $x_0 \in\,]p, g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in\,]p, g[$ holds $f'(x_0) < 0$. Then

(i) $f \restriction]p, g[$ is one-to-one,

(ii) $(f \restriction]p, g[)^{-1}$ is differentiable on $\mathrm{dom}((f \restriction]p, g[)^{-1})$,

(iii) for every x_0 such that $x_0 \in \mathrm{dom}((f\upharpoonright]p,g[)^{-1})$ holds $((f\upharpoonright]p,g[)^{-1})'(x_0) = \frac{1}{f'((f\upharpoonright]p,g[)^{-1}(x_0))}$.

(49) Suppose f is differentiable in x_0. Given h, c. Suppose $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h+c) \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}(-h+c) \subseteq \mathrm{dom}\, f$. Then $(2h)^{-1}(f \cdot (c+h) - f \cdot (c-h))$ is convergent and $\lim((2h)^{-1}(f \cdot (c+h) - f \cdot (c-h))) = f'(x_0)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[6] Jarosł aw Kotowicz. Monotonic and continuous real function. *Formalized Mathematics*, 2(**3**):403–405, 1991.
[7] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[8] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[9] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.
[10] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[11] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.
[12] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[13] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[15] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[16] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[17] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[18] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[21] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received January 10, 1991