FORMALIZED MATHEMATICS
Vol.2, No.3, May–August 1991
Université Catholique de Louvain

A Construction of Analytical Ordered Trapezium Spaces [1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We define, in a given real linear space, the midpoint operation on vectors and, with the help of the notions of directed parallelism of vectors and orthogonality of vectors, we define the relation of directed trapezium. We consider structures being enrichments of affine structures by a one binary operation, together with a function which assigns to every such structure its "affine" reduct. Theorems concerning midpoint operation and trapezium relation are proved, which enables us to introduce an abstract notion of (regular in fact) ordered trapezium space with midpoint, ordered trapezium space, and (unordered) trapezium space.

MML Identifier: ANALTRAP.

The articles [11], [2], [4], [3], [13], [9], [12], [6], [7], [10], [8], [1], and [5] provide the notation and terminology for this paper. For simplicity we follow the rules: V will denote a real linear space, u, u_1, u_2, v, v_1, v_2, w, y will denote vectors of V, a, b will denote real numbers, and x, z will be arbitrary. Let us consider V, u, v, u_1, v_1. The predicate $u,v \parallel u_1,v_1$ is defined as follows:

(Def.1) $u,v \upuparrows u_1,v_1$ or $u,v \upuparrows v_1,u_1$.

The following propositions are true:

(1) If w, y span the space, then $\mathrm{OASpace}\,V$ is an ordered affine space.

(2) For all elements p, q, p_1, q_1 of the points of $\mathrm{OASpace}\,V$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p,q \upuparrows p_1,q_1$ if and only if $u,v \upuparrows u_1,v_1$.

(3) If w, y span the space, then for all elements p, q, p_1, q_1 of the points of $\Lambda(\mathrm{OASpace}\,V)$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p,q \parallel p_1,q_1$ if and only if $u,v \parallel u_1,v_1$.

[1]Supported by RPBP.III-24.C2

ISSN 0777–4028

(4) If w, y span the space, then for all elements p, q, p_1, q_1 of the points of $\mathbf{AMSp}(V, w, y)$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p, q \parallel p_1, q_1$ if and only if $u, v \parallel u_1, v_1$.

Let us consider V, u, v. The functor $u\#v$ yielding a vector of V is defined by:

(Def.2) $u\#v + u\#v = u + v$.

One can prove the following propositions:

(5) $u\#u = u$.

(6) $u\#v = v\#u$.

(7) There exists y such that $u\#y = w$.

(8) $u\#u_1\#(v\#v_1) = u\#v\#(u_1\#v_1)$.

(9) If $u\#y = u\#w$, then $y = w$.

(10) $u, v \upuparrows y\#u, y\#v$.

(11) $u, v \parallel w\#u, w\#v$.

(12) $2 \cdot (u\#v - u) = v - u$ and $2 \cdot (v - u\#v) = v - u$.

(13) $u, u\#v \upuparrows u\#v, v$.

(14) $u, v \upuparrows u, u\#v$ and $u, v \upuparrows u\#v, v$.

(15) If $u, y \upuparrows y, v$, then $u\#y, y \upuparrows y, y\#v$.

(16) If $u, v \upuparrows u_1, v_1$, then $u, v \upuparrows u\#u_1, v\#v_1$.

Let us consider V, w, y, u, u_1, v, v_1. We say that u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y if and only if:

(Def.3) $u, u_1 \upuparrows v, v_1$ and u, u_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y and v, v_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y.

We now state a number of propositions:

(17) If w, y span the space, then u, u and v, v form a directed trapezium w.r.t. w, y.

(18) If w, y span the space, then u, v and u, v form a directed trapezium w.r.t. w, y.

(19) If u, v and v, u form a directed trapezium w.r.t. w, y, then $u = v$.

(20) If w, y span the space and v_1, u and u, v_2 form a directed trapezium w.r.t. w, y, then $v_1 = u$ and $u = v_2$.

(21) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y and u, v and u_2, v_2 form a directed trapezium w.r.t. w, y and $u \neq v$, then u_1, v_1 and u_2, v_2 form a directed trapezium w.r.t. w, y.

(22) If w, y span the space, then there exists a vector t of V such that u, v and u_1, t form a directed trapezium w.r.t. w, y or u, v and t, u_1 form a directed trapezium w.r.t. w, y.

(23) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u_1, v_1 and u, v form a directed trapezium w.r.t. w, y.

(24) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then v, u and v_1, u_1 form a directed trapezium w.r.t. w, y.

(25) If w, y span the space and v, u_1 and v, u_2 form a directed trapezium w.r.t. w, y, then $u_1 = u_2$.

(26) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y and u, v and u_1, v_2 form a directed trapezium w.r.t. w, y, then $u = v$ or $v_1 = v_2$.

(27) If w, y span the space and $u \neq u_1$ and u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y but u, u_1 and v, v_2 form a directed trapezium w.r.t. w, y or u, u_1 and v_2, v form a directed trapezium w.r.t. w, y, then $v_1 = v_2$.

(28) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u, v and $u\#u_1$, $v\#v_1$ form a directed trapezium w.r.t. w, y.

(29) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u, v and $u\#v_1$, $v\#u_1$ form a directed trapezium w.r.t. w, y or u, v and $v\#u_1$, $u\#v_1$ form a directed trapezium w.r.t. w, y.

(30) Let u, u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Then if w, y span the space and $u = u_1\#t_1$ and $u = u_2\#t_2$ and $u = v_1\#w_1$ and $u = v_2\#w_2$ and u_1, u_2 and v_1, v_2 form a directed trapezium w.r.t. w, y, then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^1_{w,y}(u)$ yielding a real number is defined as follows:

(Def.4) there exists b such that $u = \pi^1_{w,y}(u) \cdot w + b \cdot y$.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^2_{w,y}(u)$ yields a real number and is defined by:

(Def.5) there exists a such that $u = a \cdot w + \pi^2_{w,y}(u) \cdot y$.

Let us consider V, w, y, u, v. Let us assume that w, y span the space. The functor $u \cdot_{w,y} v$ yields a real number and is defined as follows:

(Def.6) $u \cdot_{w,y} v = \pi^1_{w,y}(u) \cdot \pi^1_{w,y}(v) + \pi^2_{w,y}(u) \cdot \pi^2_{w,y}(v)$.

We now state a number of propositions:

(31) If w, y span the space, then for all u, v holds $u \cdot_{w,y} v = v \cdot_{w,y} u$.

(32) Suppose w, y span the space. Given u, v, v_1. Then

(i) $u \cdot_{w,y} (v + v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$,

(ii) $u \cdot_{w,y} (v - v_1) = u \cdot_{w,y} v - u \cdot_{w,y} v_1$,

(iii) $(v - v_1) \cdot_{w,y} u = v \cdot_{w,y} u - v_1 \cdot_{w,y} u$,

(iv) $(v + v_1) \cdot_{w,y} u = v \cdot_{w,y} u + v_1 \cdot_{w,y} u$.

(33) Suppose w, y span the space. Let u, v be vectors of V. Let a be a real number. Then

(i) $(a \cdot u) \cdot_{w,y} v = a \cdot u \cdot_{w,y} v$,

(ii) $u \cdot_{w,y} (a \cdot v) = a \cdot u \cdot_{w,y} v$,

(iii) $(a \cdot u) \cdot_{w,y} v = u \cdot_{w,y} v \cdot a$,

(iv) $u \cdot_{w,y} (a \cdot v) = u \cdot_{w,y} v \cdot a$.

(34) If w, y span the space, then for all vectors u, v of V holds u, v are orthogonal w.r.t. w, y if and only if $u \cdot_{w,y} v = 0$.

(35) If w, y span the space, then for all vectors u, v, u_1, v_1 of V holds u, v, u_1 and v_1 are orthogonal w.r.t. w, y if and only if $(v-u) \cdot_{w,y} (v_1 - u_1) = 0$.

(36) If w, y span the space, then for all vectors u, v, v_1 of V holds $2 \cdot u \cdot_{w,y} (v \# v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$.

(37) If w, y span the space, then for all vectors u, v of V such that $u \neq v$ holds $(u - v) \cdot_{w,y} (u - v) \neq 0$.

(38) Suppose w, y span the space. Let p, q, u, v, v' be vectors of V. Let A be a real number. Suppose that

(i) p, q and u, v form a directed trapezium w.r.t. w, y,

(ii) $p \neq q$,

(iii) $A = ((p - q) \cdot_{w,y} (p + q) - 2 \cdot (p - q) \cdot_{w,y} u) \cdot (p - q) \cdot_{w,y} (p - q)^{-1}$,

(iv) $v' = u + A \cdot (p - q)$.

Then $v = v'$.

(39) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Then if $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and $u_1, u_2 \upuparrows v_1, v_2$, then $t_1, t_2 \upuparrows w_1, w_2$.

(40) Suppose w, y span the space. Then for all vectors u, u', u_1, u_2, v_1, t_1, t_2, w_1 of V such that $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and $v_1 = u_1 \# u_2$ holds $w_1 = t_1 \# t_2$.

(41) If w, y span the space, then for all vectors u, u', u_1, u_2, t_1, t_2 of V such that $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y holds u, u' and $u_1 \# u_2$, $t_1 \# t_2$ form a directed trapezium w.r.t. w, y.

(42) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and u_1, u_2, v_1 and v_2 are orthogonal w.r.t. w, y. Then t_1, t_2, w_1 and w_2 are orthogonal w.r.t. w, y.

(43) Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose w, y span the space and $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and u_1, u_2 and v_1, v_2 form

a directed trapezium w.r.t. w, y. Then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V, w, y. The
directed trapezium relation defined over V in the basis w, y
yielding a binary relation on [: the vectors of V, the vectors of V :] is defined as follows:

(Def.7) $\langle x, z\rangle \in$ the directed trapezium relation defined over V in the basis w, y if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y.

The following proposition is true

(44) If w, y span the space, then $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in$ the directed trapezium relation defined over V in the basis w, y if and only if u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

Let us consider V. The midpoint operation in V yields a binary operation on the vectors of V and is defined as follows:

(Def.8) for all u, v holds (the midpoint operation in V)$(u, v) = u\#v$.

We consider affine midpoint structures which are systems
$\langle$points, a midpoint operation, a congruence$\rangle$,
where the points constitute a non-empty set, the midpoint operation is a binary operation on the points, and the congruence is a binary relation on [: the points, the points :].

Let us consider V, w, y. Let us assume that w, y span the space. The directed trapezium space defined over V in the basis w, y yielding a affine midpoint structure is defined as follows:

(Def.9) the directed trapezium space defined over V in the basis $w, y = \langle$the vectors of V, the midpoint operation in V, the directed trapezium relation defined over V in the basis $w, y\rangle$.

The following proposition is true

(45) For all V, w, y such that w, y span the space holds the directed trapezium space defined over V in the basis $w, y = \langle$the vectors of V, the midpoint operation in V, the directed trapezium relation defined over V in the basis $w, y\rangle$.

Let A_1 be a affine midpoint structure. The affine reduct of A_1 yielding an affine structure is defined by:

(Def.10) the affine reduct of $A_1 = \langle$ the points of A_1, the congruence of $A_1\rangle$.

Let A_1 be a affine midpoint structure, and let a, b, c, d be elements of the points of A_1. The predicate $a, b \top^{>} c, d$ is defined by:

(Def.11) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of A_1.

Let A_1 be a affine midpoint structure, and let a, b be elements of the points of A_1. The functor $a\#b$ yielding an element of the points of A_1 is defined by:

(Def.12) $a\#b =$ (the midpoint operation of A_1)(a, b).

In the sequel a, b, a_1, b_1 denote elements of the points of the directed trapezium space defined over V in the basis w, y.

We now state three propositions:

(46) If w, y span the space, then for an arbitrary x holds x is an element of the points of the directed trapezium space defined over V in the basis w, y if and only if x is a vector of V.

(47) If w, y span the space and $u = a$ and $v = b$ and $u_1 = a_1$ and $v_1 = b_1$, then $a, b \top^{>} a_1, b_1$ if and only if u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

(48) If w, y span the space and $u = a$ and $v = b$, then $u\#v = a\#b$.

A affine midpoint structure is called an ordered midpoint trapezium space if it satisfies the condition (Def.13).

(Def.13) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of the points of it . Then

(i) $a\#b = b\#a$,

(ii) $a\#a = a$,

(iii) $a\#b\#(c\#d) = a\#c\#(b\#d)$,

(iv) there exists an element p of the points of it such that $p\#a = b$,

(v) if $a\#b = a\#c$, then $b = c$,

(vi) if $a, b \top^{>} c, d$, then $a, b \top^{>} a\#c, b\#d$,

(vii) if $a, b \top^{>} c, d$, then $a, b \top^{>} a\#d, b\#c$ or $a, b \top^{>} b\#c, a\#d$,

(viii) if $a, b \top^{>} c, d$ and $a\#a_1 = p$ and $b\#b_1 = p$ and $c\#c_1 = p$ and $d\#d_1 = p$, then $a_1, b_1 \top^{>} c_1, d_1$,

(ix) if $p \neq q$ and $p, q \top^{>} a, a_1$ and $p, q \top^{>} b, b_1$ and $p, q \top^{>} c, c_1$ and $p, q \top^{>} d, d_1$ and $a, b \top^{>} c, d$, then $a_1, b_1 \top^{>} c_1, d_1$,

(x) if $a, b \top^{>} b, c$, then $a = b$ and $b = c$,

(xi) if $a, b \top^{>} a_1, b_1$ and $a, b \top^{>} c_1, d_1$ and $a \neq b$, then $a_1, b_1 \top^{>} c_1, d_1$,

(xii) if $a, b \top^{>} c, d$, then $c, d \top^{>} a, b$ and $b, a \top^{>} d, c$,

(xiii) there exists an element d of the points of it such that $a, b \top^{>} c, d$ or $a, b \top^{>} d, c$,

(xiv) if $a, b \top^{>} c, p$ and $a, b \top^{>} c, q$, then $a = b$ or $p = q$.

One can prove the following proposition

(49) If w, y span the space, then the directed trapezium space defined over V in the basis w, y is an ordered midpoint trapezium space.

An affine structure is called an ordered trapezium space if it satisfies the condition (Def.14).

(Def.14) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of the points of it . Then

(i) if $a, b \parallel b, c$, then $a = b$ and $b = c$,

(ii) if $a, b \parallel a_1, b_1$ and $a, b \parallel c_1, d_1$ and $a \neq b$, then $a_1, b_1 \parallel c_1, d_1$,

(iii) if $a, b \parallel c, d$, then $c, d \parallel a, b$ and $b, a \parallel d, c$,

(iv) there exists an element d of the points of it such that $a, b \parallel c, d$ or $a, b \parallel d, c$,

(v) if $a, b \parallel c, p$ and $a, b \parallel c, q$, then $a = b$ or $p = q$.

Let M_1 be an ordered midpoint trapezium space. Then the affine reduct of M_1 is an ordered trapezium space.

We follow a convention: O_1 denotes an ordered trapezium space, a, b, c, d denote elements of the points of O_1, and a', b', c', d' denote elements of the points of $\Lambda(O_1)$. We now state two propositions:

(50) For an arbitrary x holds x is an element of the points of O_1 if and only if x is an element of the points of $\Lambda(O_1)$.

(51) If $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$, then $a', b' \parallel c', d'$ if and only if $a, b \parallel c, d$ or $a, b \parallel d, c$.

An affine structure is called a trapezium space if it satisfies the condition (Def.15).

(Def.15) Let a', b', c', d', p', q' be elements of the points of it . Then

(i) $a', b' \parallel b', a'$,

(ii) if $a', b' \parallel c', d'$ and $a', b' \parallel c', q'$, then $a' = b'$ or $d' = q'$,

(iii) if $p' \neq q'$ and $p', q' \parallel a', b'$ and $p', q' \parallel c', d'$, then $a', b' \parallel c', d'$,

(iv) if $a', b' \parallel c', d'$, then $c', d' \parallel a', b'$,

(v) there exists an element x' of the points of it such that $a', b' \parallel c', x'$.

Let O_1 be an ordered trapezium space. Then $\Lambda(O_1)$ is a trapezium space.

An affine structure is regular if it satisfies the condition (Def.16).

(Def.16) Let p, q, a, a_1, b, b_1, c, c_1, d, d_1 be elements of the points of it . Then if $p \neq q$ and $p, q \parallel a, a_1$ and $p, q \parallel b, b_1$ and $p, q \parallel c, c_1$ and $p, q \parallel d, d_1$ and $a, b \parallel c, d$, then $a_1, b_1 \parallel c_1, d_1$.

Let M_1 be an ordered midpoint trapezium space. Then the affine reduct of M_1 is an regular ordered trapezium space.

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Received October 29, 1990