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Graphs

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Summary. Definitions of graphs are introduced and their basic properties are proved. The following notions related to graph theory are introduced: Subgraph, Finite graph, Chain and oriented chain - as a finite sequence of edges, Path and oriented path - as a finite sequence of different edges, Cycle and oriented cycle, Incidency of graph's vertices, A sum of two graphs, A degree of a vertice, A set of all subgraphs of a graph. Many ideas in this article have been taken from [12].

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The terminology and notation used in this paper are introduced in the following papers: [10], [4], [5], [3], [9], [7], [6], [1], [8], [2], and [11]. We adopt the following convention: x, y, v will be arbitrary and n, m will be natural numbers. We consider multi graph structures which are systems

$\langle$vertices, edges, a source, a target$\rangle$,

where the vertices, the edges constitute a set and the source, the target are a function from the edges into the vertices.

A multi graph structure is said to be a graph if:

(Def.1) the vertices of it is a non-empty set.

In the sequel G, G_1, G_2, G_3 are graphs. Let us consider G_1, G_2. Let us assume that the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. The functor $G_1 \cup G_2$ yielding a graph is defined by the conditions (Def.2).

(Def.2) (i) The vertices of $G_1 \cup G_2$ = (the vertices of G_1)$\cup$ the vertices of G_2,

(ii) the edges of $G_1 \cup G_2$ = (the edges of G_1)$\cup$ the edges of G_2,

(iii) for every v such that $v \in$ the edges of G_1 holds (the source of $G_1 \cup G_2$)(v) = (the source of G_1)(v) and (the target of $G_1 \cup G_2$)(v) = (the target of G_1)(v),

(iv) for every v such that $v \in$ the edges of G_2 holds (the source of $G_1 \cup G_2$)(v) = (the source of G_2)(v) and (the target of $G_1 \cup G_2$)(v) = (the target of G_2)(v).

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Let G, G_1, G_2 be graphs. We say that G is a sum of G_1 and G_2 if and only if:

(Def.3) the target of $G_1 \approx$ the target of G_2 and the source of $G_1 \approx$ the source of G_2 and $G = G_1 \cup G_2$.

We now define five new attributes. A graph is oriented if:

(Def.4) for all x, y such that $x \in$ the edges of it and $y \in$ the edges of it and (the source of it)$(x) =$ (the source of it)(y) and (the target of it)$(x) =$ (the target of it)(y) holds $x = y$.

A graph is non-multi if it satisfies the condition (Def.5).

(Def.5) Given x, y. Suppose $x \in$ the edges of it and $y \in$ the edges of it but (the source of it)$(x) =$ (the source of it)(y) and (the target of it)$(x) =$ (the target of it)(y) or (the source of it)$(x) =$ (the target of it)(y) and (the source of it)$(y) =$ (the target of it)(x). Then $x = y$.

A graph is simple if:

(Def.6) for no x holds $x \in$ the edges of it and (the source of it)$(x) =$ (the target of it)(x).

A graph is connected if:

(Def.7) for no graphs G_1, G_2 holds (the vertices of G_1)$\cap$ the vertices of $G_2 = \emptyset$ and it is a sum of G_1 and G_2.

A multi graph structure is finite if:

(Def.8) the vertices of it is finite and the edges of it is finite.

In the sequel x, y will denote elements of the vertices of G. Let us consider G, x, y, v. We say that v joins x with y if and only if:

(Def.9) (the source of G)$(v) = x$ and (the target of G)$(v) = y$ or (the source of G)$(v) = y$ and (the target of G)$(v) = x$.

Let us consider G, and let x, y be elements of the vertices of G. We say that x and y are incydent if and only if:

(Def.10) there exists arbitrary v such that $v \in$ the edges of G and v joins x with y.

Let G be a graph. A finite sequence is called a chain of G if it satisfies the conditions (Def.11).

(Def.11) (i) For every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n) \in$ the edges of G,

(ii) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $\mathrm{it}(n)$ joins x' with y'.

Let G be a graph. A chain of G is said to be an oriented chain of G if:

(Def.12) for every n such that $1 \leq n$ and $n < \operatorname{len} \mathrm{it}$ holds (the source of G)$(\mathrm{it}(n+1)) =$ (the target of G)$(\mathrm{it}(n))$.

Let G be a graph. A chain of G is said to be a path of G if:

(Def.13) for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n) \neq \mathrm{it}(m)$.

Let G be a graph. An oriented chain of G is said to be an oriented path of G if:

(Def.14) it is a path of G.

Let G be a graph. A path of G is said to be a cycle of G if it satisfies the condition (Def.15).

(Def.15) There exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $\mathrm{it}(n)$ joins x' with y' and $p(1) = p(\operatorname{len} p)$.

Let G be a graph. An oriented path of G is called an oriented cycle of G if:

(Def.16) it is a cycle of G.

Let G be a graph. A graph is said to be a subgraph of G if it satisfies the conditions (Def.17).

(Def.17) (i) The vertices of it $\subseteq$ the vertices of G,

(ii) the edges of it $\subseteq$ the edges of G,

(iii) for every v such that $v \in$ the edges of it holds (the source of it)$(v) =$ (the source of $G)(v)$ and (the target of it)$(v) =$ (the target of $G)(v)$ and (the source of $G)(v) \in$ the vertices of it and (the target of $G)(v) \in$ the vertices of it.

We now define two new functors. Let G be an finite graph. The number of vertices of G yielding a natural number is defined by:

(Def.18) the number of vertices of $G =$ card (the vertices of G).

The number of edges of G yielding a natural number is defined by:

(Def.19) the number of edges of $G =$ card (the edges of G).

We now define two new functors. Let G be an finite graph, and let x be an element of the vertices of G. The functor $\mathrm{EdgIn}(x)$ yields a natural number and is defined as follows:

(Def.20) there exists a set X such that for an arbitrary z holds $z \in X$ if and only if $z \in$ the edges of G and (the target of $G)(z) = x$ and $\mathrm{EdgIn}(x) = \operatorname{card} X$.

The functor $\mathrm{EdgOut}(x)$ yielding a natural number is defined by:

(Def.21) there exists a set X such that for an arbitrary z holds $z \in X$ if and only if $z \in$ the edges of G and (the source of $G)(z) = x$ and $\mathrm{EdgOut}(x) = \operatorname{card} X$.

Let G be an finite graph, and let x be an element of the vertices of G. The degree of x yields a natural number and is defined by:

(Def.22) the degree of $x = \mathrm{EdgIn}(x) + \mathrm{EdgOut}(x)$.

Let G_1, G_2 be graphs. The predicate $G_1 \subseteq G_2$ is defined by:

(Def.23) G_1 is a subgraph of G_2.

Let G be a graph. The functor 2^G yields a set and is defined by:

(Def.24) for an arbitrary x holds $x \in 2^G$ if and only if x is a subgraph of G.

The scheme *GraphSeparation* deals with a graph $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a set X such that for an arbitrary x holds $x \in X$ if and only if x is a subgraph of $\mathcal{A}$ and $\mathcal{P}[x]$

for all values of the parameters.

Next we state a number of propositions:

(1) For every graph G holds dom (the source of G) = the edges of G and dom (the target of G) = the edges of G and rng (the source of G) $\subseteq$ the vertices of G and rng (the target of G) $\subseteq$ the vertices of G.

(2) For every element x of the vertices of G holds $x \in$ the vertices of G.

(3) For an arbitrary v such that $v \in$ the edges of G holds (the source of $G)(v) \in$ the vertices of G and (the target of $G)(v) \in$ the vertices of G.

(4) For every chain p of G holds $p \restriction \mathrm{Seg}\, n$ is a chain of G.

(5) If $G_1 \subseteq G$, then graph (the source of G_1) $\subseteq$ graph (the source of G) and graph (the target of G_1) $\subseteq$ graph (the target of G).

(6) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then graph (the source of $G_1 \cup G_2$) = graph (the source of G_1) $\cup$ graph (the source of G_2) and graph (the target of $G_1 \cup G_2$) = graph (the target of G_1) $\cup$ graph (the target of G_2).

(7) $G = G \cup G$.

(8) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \cup G_2 = G_2 \cup G_1$.

(9) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and the source of $G_1 \approx$ the source of G_3 and the target of $G_1 \approx$ the target of G_3 and the source of $G_2 \approx$ the source of G_3 and the target of $G_2 \approx$ the target of G_3, then $G_1 \cup G_2 \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(10) If G is a sum of G_1 and G_2, then G is a sum of G_2 and G_1.

(11) G is a sum of G and G.

(12) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 = G_2 \cup G_1$.

(13) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ and $G_3 \subseteq G$, then $G_1 \cup G_2 \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(14) $G \subseteq G$.

(15) For all subgraphs H_1, H_2 of G such that the vertices of H_1 = the vertices of H_2 and the edges of H_1 = the edges of H_2 holds $H_1 = H_2$.

(16) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$, then $G_1 = G_2$.

(17) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_3$, then $G_1 \subseteq G_3$.

(18) If G is a sum of G_1 and G_2, then $G_1 \subseteq G$ and $G_2 \subseteq G$.

(19) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(20) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(21) If $G_1 \subseteq G_3$ and $G_2 \subseteq G_3$ and G is a sum of G_1 and G_2, then $G \subseteq G_3$.

(22) If $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 \subseteq G$.

(23) If $G_1 \subseteq G_2$, then $G_1 \cup G_2 = G_2$ and $G_2 \cup G_1 = G_2$.

(24) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 but $G_1 \cup G_2 = G_2$ or $G_2 \cup G_1 = G_2$, then $G_1 \subseteq G_2$.

(25) If G_2 is a sum of G_1 and G_2 or G_2 is a sum of G_2 and G_1, then $G_1 \subseteq G_2$.

(26) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ but $G_2 = G_1 \cup G_2$ or $G_2 = G_2 \cup G_1$, then $G_1 \subseteq G_2$.

(27) For every oriented graph G such that $G_1 \subseteq G$ holds G_1 is oriented.

(28) For every non-multi graph G such that $G_1 \subseteq G$ holds G_1 is non-multi.

(29) For every simple graph G such that $G_1 \subseteq G$ holds G_1 is simple.

(30) $G_1 \in 2^G$ if and only if $G_1 \subseteq G$.

(31) $G \in 2^G$.

We now state several propositions:

(32) $G_1 \subseteq G_2$ if and only if $2^{G_1} \subseteq 2^{G_2}$.

(33) $2^G \neq \emptyset$.

(34) $\{G\} \subseteq 2^G$.

(35) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and $2^{G_1 \cup G_2} \subseteq 2^{G_1} \cup 2^{G_2}$, then $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$.

(36) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $2^{G_1} \cup 2^{G_2} \subseteq 2^{G_1 \cup G_2}$.

(37) If $G_1 \in 2^G$ and $G_2 \in 2^G$, then $G_1 \cup G_2 \in 2^G$.

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