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Fanoian, Pappian and Desarguesian Affine Spaces

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Summary. We introduce basic types of affine spaces such as Desarguesian, Fanoian, Pappian, and translation affine and ordered affine spaces and we prove that suitably chosen analytically defined affine structures satify the required properties.

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The articles [6], [1], [4], [5], [2], and [3] provide the notation and terminology for this paper. Let O_1 be an ordered affine space. Then $\Lambda(O_1)$ is an affine space.

Let O_1 be an ordered affine plane. Then $\Lambda(O_1)$ is an affine plane.

We now state several propositions:

(1) There exists a real linear space V and there exist vectors u, v of V such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$.

(2) For every ordered affine space O_1 and for an arbitrary x holds x is an element of the points of O_1 if and only if x is an element of the points of $\Lambda(O_1)$ but x is a subset of the points of O_1 if and only if x is a subset of the points of $\Lambda(O_1)$.

(3) For every ordered affine space O_1 and for all elements a, b, c of the points of O_1 and for all elements a', b', c' of the points of $\Lambda(O_1)$ such that $a = a'$ and $b = b'$ and $c = c'$ holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(a', b', c')$.

(4) For every real linear space V and for an arbitrary x holds x is an element of the points of OASpace V if and only if x is a vector of V.

(5) Let V be a real linear space. Then for every ordered affine space O_1 such that $O_1 = \text{OASpace}\,V$ for all elements a, b, c, d of the points of O_1

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and for all vectors u, v, w, y of V such that $a = u$ and $b = v$ and $c = w$ and $d = y$ holds $a, b \parallel c, d$ if and only if $u, v \parallel w, y$.

(6) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ there exist vectors u, v of V such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$.

Let A_1 be an affine space. We say that A_1 satisfies **PAP'** if and only if the condition (Def.1) is satisfied.

(Def.1) Let M, N be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,
(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $a' \in N$,
(xvi) $b' \in N$,
(xvii) $c' \in N$,
(xviii) $a, b' \parallel b, a'$,
(xix) $b, c' \parallel c, b'$.

Then $a, c' \parallel c, a'$.

Let A_1 be an affine space. We say that A_1 satisfies **DES'** if and only if the condition (Def.2) is satisfied.

(Def.2) Let A, P, C be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,

(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies **TDES'** if and only if the condition (Def.3) is satisfied.

(Def.3) Let K be a subset of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that
(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $a, b \parallel a', b'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies **des'** if and only if the condition (Def.4) is satisfied.

(Def.4) Let A, P, C be subsets of the points of A_1. Let a, b, c, a', b', c' be elements of the points of A_1. Suppose that
(i) $A \parallel P$,
(ii) $A \parallel C$,
(iii) $a \in A$,
(iv) $a' \in A$,
(v) $b \in P$,
(vi) $b' \in P$,
(vii) $c \in C$,
(viii) $c' \in C$,
(ix) A is a line,
(x) P is a line,
(xi) C is a line,
(xii) $A \neq P$,
(xiii) $A \neq C$,
(xiv) $a, b \parallel a', b'$,

(xv) $a, c \parallel a', c'$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies Fano Axiom if and only if:

(Def.5) for all elements a, b, c, d of the points of A_1 such that $a, b \parallel c, d$ and $a, c \parallel b, d$ and $a, d \parallel b, c$ holds $a, b \parallel a, c$.

One can prove the following propositions:

(7) For every affine plane A_1 holds A_1 satisfies **PAP** if and only if A_1 satisfies **PAP'**.

(8) For every affine plane A_1 holds A_1 satisfies **DES** if and only if A_1 satisfies **DES'**.

(9) For every affine plane A_1 holds A_1 satisfies **TDES** if and only if A_1 satisfies **TDES'**.

(10) For every affine plane A_1 holds A_1 satisfies **des** if and only if A_1 satisfies **des'**.

An affine space is Pappian if:

(Def.6) it satisfies **PAP'**.

An affine space is Desarguesian if:

(Def.7) it satisfies **DES'**.

An affine space is Moufangian if:

(Def.8) it satisfies **TDES'**.

An affine space is translation if:

(Def.9) it satisfies **des'**.

An affine space is Fanoian if:

(Def.10) it satisfies Fano Axiom.

An ordered affine space is Pappian if:

(Def.11) Λ(it) satisfies **PAP'**.

An ordered affine space is Desarguesian if:

(Def.12) Λ(it) satisfies **DES'**.

An ordered affine space is Moufangian if:

(Def.13) Λ(it) satisfies **TDES'**.

An ordered affine space is translation if:

(Def.14) Λ(it) satisfies **des'**.

Let O_1 be an ordered affine space. We say that O_1 satisfies **DES** if and only if the condition (Def.15) is satisfied.

(Def.15) Let o, a, b, c, a_1, b_1, c_1 be elements of the points of O_1. Then if $o, a \upuparrows o, a_1$ and $o, b \upuparrows o, b_1$ and $o, c \upuparrows o, c_1$ and not $\mathbf{L}(o, a, b)$ and not $\mathbf{L}(o, a, c)$ and $a, b \upuparrows a_1, b_1$ and $a, c \upuparrows a_1, c_1$, then $b, c \upuparrows b_1, c_1$.

Let O_1 be an ordered affine space. We say that O_1 satisfies $\mathbf{DES_1}$ if and only if the condition (Def.16) is satisfied.

(Def.16) Let o, a, b, c, a_1, b_1, c_1 be elements of the points of O_1. Then if $a, o \upuparrows o, a_1$ and $b, o \upuparrows o, b_1$ and $c, o \upuparrows o, c_1$ and not $\mathbf{L}(o, a, b)$ and not $\mathbf{L}(o, a, c)$ and $a, b \upuparrows b_1, a_1$ and $a, c \upuparrows c_1, a_1$, then $b, c \upuparrows c_1, b_1$.

One can prove the following propositions:

(11) For every ordered affine space O_1 such that O_1 satisfies $\mathbf{DES_1}$ holds O_1 satisfies **DES**.

(12) For every ordered affine space O_1 and for all elements o, a, b, a', b' of the points of O_1 such that not $\mathbf{L}(o, a, b)$ and $a, o \upuparrows o, a'$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ holds $b, o \upuparrows o, b'$ and $a, b \upuparrows b', a'$.

(13) For every ordered affine space O_1 and for all elements o, a, b, a', b' of the points of O_1 such that not $\mathbf{L}(o, a, b)$ and $o, a \upuparrows o, a'$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ holds $o, b \upuparrows o, b'$ and $a, b \upuparrows a', b'$.

(14) For every ordered affine space O_2 such that O_2 satisfies $\mathbf{DES_1}$ holds $\Lambda(O_2)$ satisfies **DES'**.

(15) Let V be a real linear space. Let o, u, v, u_1, v_1 be vectors of V. Let r be a real number. Suppose $o - u = r \cdot (u_1 - o)$ and $r \neq 0$ and $o, v \parallel o, v_1$ and $o, u \nparallel o, v$ and $u, v \parallel u_1, v_1$. Then $v_1 = u_1 + (-r)^{-1} \cdot (v - u)$ and $v_1 = o + (-r)^{-1} \cdot (v - o)$ and $v - u = (-r) \cdot (v_1 - u_1)$.

(16) For every real number r such that $r \neq 0$ holds $(-r)^{-1} = -r^{-1}$.

(17) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds O_1 satisfies $\mathbf{DES_1}$.

(18) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds O_1 satisfies $\mathbf{DES_1}$ and O_1 satisfies **DES**.

(19) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **PAP'**.

(20) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **DES'**.

(21) For every affine space A_1 such that A_1 satisfies **DES'** holds A_1 satisfies **TDES'**.

(22) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **TDES'**.

(23) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **des'**.

(24) For every ordered affine space O_1 holds $\Lambda(O_1)$ satisfies Fano Axiom.

Let O_1 be an ordered affine space. Then $\Lambda(O_1)$ is an Fanoian affine space.

Let O_1 be a Pappian ordered affine space. Then $\Lambda(O_1)$ is a Pappian Fanoian affine space.

Let O_1 be a Desarguesian ordered affine space. Then $\Lambda(O_1)$ is an Desarguesian Fanoian affine space.

Let O_1 be a Moufangian ordered affine space. Then $\Lambda(O_1)$ is an Moufangian Fanoian affine space.

Let O_1 be a translation ordered affine space. Then $\Lambda(O_1)$ is a translation Fanoian affine space.

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