FORMALIZED MATHEMATICS
Vol.2, No.3, May–August 1991
Université Catholique de Louvain

Preliminaries to the Lambek Calculus

Wojciech Zielonka[1]
Adam Mickiewicz University
Poznań

Summary. Some preliminary facts concerning completeness and decidability problems for the Lambek calculus [13] are proved as well as some theses and derived rules of the calculus itself.

MML Identifier: `PRELAMB`.

The articles [16], [7], [9], [10], [18], [6], [8], [12], [17], [15], [14], [5], [1], [11], [2], [3], and [4] provide the terminology and notation for this paper. We consider structures of the type algebra which are systems

⟨types, a left quotient, a right quotient, a inner product⟩,

where the types constitute a non-empty set and the left quotient, the right quotient, the inner product are a binary operation on the types.

Let s be a structure of the type algebra. A type of s is an element of the types of s.

We adopt the following rules: s will denote a structure of the type algebra, T, X, Y will denote finite sequences of elements of the types of s, and x, y, z will denote types of s. We now define three new functors. Let us consider s, x, y. The functor $x \setminus y$ yields a type of s and is defined by:

(Def.1) $x \setminus y =$ (the left quotient of s)(x, y).

The functor x/y yields a type of s and is defined as follows:

(Def.2) $x/y =$ (the right quotient of s)(x, y).

The functor $x \cdot y$ yields a type of s and is defined by:

(Def.3) $x \cdot y =$ (the inner product of s)(x, y).

Let T_1 be a tree, and let v be an element of T_1. The branch degree of v is defined by:

(Def.4) the branch degree of $v = \mathrm{card}\,\mathrm{succ}\,v$.

[1]This paper was written during author's visit at the Warsaw University (Białystok) in Winter 1991.

© 1991 Fondation Philippe le Hodey
ISSN 0777–4028

Let us consider s. A preproof of s is a tree decorated by $[\![\,[\![\,(\text{the types of } s)^*, \text{the types of } s\,]\!], \mathbb{N}\,]\!]$.

In the sequel T_1 is a preproof of s. Let us consider s, T_1, and let v be an element of $\operatorname{dom} T_1$. We say that v is correct if and only if:

(Def.5) (i) the branch degree of $v = 0$ and there exists x such that $T_1(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 0$,

(ii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 1$,

(iii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, y \setminus x\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$ if $T_1(v)_{\mathbf{2}} = 2$,

(iv) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 3$,

(v) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 4$,

(vi) the branch degree of $v = 1$ and there exist X, x, y, Y such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 5$,

(vii) the branch degree of $v = 2$ and there exist X, Y, x, y such that $T_1(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X, x\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle Y, y\rangle$ if $T_1(v)_{\mathbf{2}} = 6$,

(viii) the branch degree of $v = 2$ and there exist T, X, Y, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 7$.

We now define three new attributes. Let us consider s. A type of s is left if:

(Def.6) there exist x, y such that it $= x \setminus y$.

A type of s is right if:

(Def.7) there exist x, y such that it $= x/y$.

A type of s is middle if:

(Def.8) there exist x, y such that it $= x \cdot y$.

Let us consider s. A type of s is primitive if:

(Def.9) neither it is left nor it is right nor it is middle.

Let us consider s, and let T_1 be a tree decorated by the types of s, and let us consider x. We say that T_1 represents x if and only if the conditions (Def.10) is satisfied.

(Def.10) (i) $\operatorname{dom} T_1$ is finite,

(ii) for every element v of $\operatorname{dom} T_1$ holds the branch degree of $v = 0$ or the branch degree of $v = 2$ but if the branch degree of $v = 0$, then $T_1(v)$ is primitive but if the branch degree of $v = 2$, then there exist y, z such that $T_1(v) = y/z$ or $T_1(v) = y \setminus z$ or $T_1(v) = y \cdot z$ but $T_1(v \frown \langle 0\rangle) = y$ and $T_1(v \frown \langle 1\rangle) = z$.

A structure of the type algebra is free if:

(Def.11) for no type x of it holds x is left right or x is left middle or x is right middle and for every type x of it there exists a tree T_1 decorated by the types of it such that for every tree T_2 decorated by the types of it holds T_2 represents x if and only if $T_1 = T_2$.

Let us consider s, x. Let us assume that s is free. The representation of x yields a tree decorated by the types of s and is defined by:

(Def.12) for every tree T_1 decorated by the types of s holds T_1 represents x if and only if the representation of $x = T_1$.

Let us consider s, and let f be a finite sequence of elements of the types of s, and let t be a type of s. Then $\langle f, t \rangle$ is an element of $[:$(the types of $s)^*$, the types of $s:]$.

Let us consider s. A preproof of s is called a proof of s if:

(Def.13) dom it is a finite tree and for every element v of dom it holds v is correct.

In the sequel p is a proof of s and v is an element of dom p. The following propositions are true:

(1) If the branch degree of $v = 1$, then $v \frown \langle 0 \rangle \in \operatorname{dom} p$.

(2) If the branch degree of $v = 2$, then $v \frown \langle 0 \rangle \in \operatorname{dom} p$ and $v \frown \langle 1 \rangle \in \operatorname{dom} p$.

(3) If $p(v)_{\mathbf{2}} = 0$, then there exists x such that $p(v)_{\mathbf{1}} = \langle \langle x \rangle, x \rangle$.

(4) If $p(v)_{\mathbf{2}} = 1$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle T, x/y \rangle$ and $p(w)_{\mathbf{1}} = \langle T \frown \langle y \rangle, x \rangle$.

(5) If $p(v)_{\mathbf{2}} = 2$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle T, y \setminus x \rangle$ and $p(w)_{\mathbf{1}} = \langle \langle y \rangle \frown T, x \rangle$.

(6) Suppose $p(v)_{\mathbf{2}} = 3$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x/y \rangle \frown T \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown Y, z \rangle$.

(7) Suppose $p(v)_{\mathbf{2}} = 4$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x \rangle \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown Y, z \rangle$.

(8) Suppose $p(v)_{\mathbf{2}} = 5$. Then there exists an element w of dom p and there exist X, x, y, Y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y \rangle \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown \langle y \rangle \frown Y, z \rangle$.

(9) Suppose $p(v)_{\mathbf{2}} = 6$. Then there exist elements w, u of dom p and there exist X, Y, x, y such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y \rangle$ and $p(w)_{\mathbf{1}} = \langle X, x \rangle$ and $p(u)_{\mathbf{1}} = \langle Y, y \rangle$.

(10) Suppose $p(v)_{\mathbf{2}} = 7$. Then there exist elements w, u of dom p and there exist T, X, Y, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle y \rangle \frown Y, z \rangle$.

(11) (i) $p(v)_{\mathbf{2}} = 0$, or
(ii) $p(v)_{\mathbf{2}} = 1$, or
(iii) $p(v)_{\mathbf{2}} = 2$, or
(iv) $p(v)_{\mathbf{2}} = 3$, or
(v) $p(v)_{\mathbf{2}} = 4$, or

(vi) $p(v)_{\mathbf{2}} = 5$, or
(vii) $p(v)_{\mathbf{2}} = 6$, or
(viii) $p(v)_{\mathbf{2}} = 7$.

We now define two new constructions. Let us consider s. A preproof of s is cut-free if:

(Def.14) for every element v of dom it holds $\mathrm{it}(v)_{\mathbf{2}} \neq 7$.

The size w.r.t. s yielding a function from the types of s into $\mathbb{N}$ is defined by:

(Def.15) for every x holds
(the size w.r.t. s)(x) = card dom(the representation of x).

Let D be a non-empty set, and let T be a finite sequence of elements of D, and let f be a function from D into $\mathbb{N}$. Then $f \cdot T$ is a finite sequence of elements of $\mathbb{R}$.

Let D be a non-empty set, and let f be a function from D into $\mathbb{N}$, and let d be an element of D. Then $f(d)$ is a natural number.

Let us consider s, and let p be a proof of s. Let us assume that s is free. The cut degree of p yields a natural number and is defined by:

(Def.16) (i) there exist T, X, Y, y, z such that $p(\varepsilon)_{\mathbf{1}} = \langle X \frown T \frown Y, z \rangle$ and $p(\langle 0 \rangle)_{\mathbf{1}} = \langle T, y \rangle$ and $p(\langle 1 \rangle)_{\mathbf{1}} = \langle X \frown \langle y \rangle \frown Y, z \rangle$ and the cut degree of p = (the size w.r.t. s)(y) + (the size w.r.t. s)$(z) + \sum$((the size w.r.t. s) $\cdot (X \frown T \frown Y)$) if $p(\varepsilon)_{\mathbf{2}} = 7$,
(ii) the cut degree of $p = 0$, otherwise.

We adopt the following convention: A denotes an non-empty set and a, a_1, a_2, b denote elements of A^*. Let us consider s, A. A function from the types of s into 2^{A^*} is said to be a model of s if it satisfies the condition (Def.17).

(Def.17) Given x, y. Then
(i) $\mathrm{it}(x \cdot y) = \{a \frown b : a \in \mathrm{it}(x) \wedge b \in \mathrm{it}(y)\}$,
(ii) $\mathrm{it}(x/y) = \{a_1 : \bigwedge_b [b \in \mathrm{it}(y) \Rightarrow a_1 \frown b \in \mathrm{it}(x)]\}$,
(iii) $\mathrm{it}(y \setminus x) = \{a_2 : \bigwedge_b [b \in \mathrm{it}(y) \Rightarrow b \frown a_2 \in \mathrm{it}(x)]\}$.

We consider type structures which are systems
⟨structures of the type algebra; a derivability⟩,
where the derivability is a non-empty relation between
(the types of the structure of the type algebra)*
and the types of the structure of the type algebra.

In the sequel s will denote a type structure and x will denote a type of s. Let us consider s, and let f be a finite sequence of elements of the types of s, and let us consider x. The predicate $f \longrightarrow x$ is defined by:

(Def.18) $\langle f, x \rangle \in$ the derivability of s.

A type structure is called a calculus of syntactic types if it satisfies the conditions (Def.19).

(Def.19) (i) For every type x of it holds $\langle x \rangle \longrightarrow x$,
(ii) for every finite sequence T of elements of the types of it and for all types x, y of it such that $T \frown \langle y \rangle \longrightarrow x$ holds $T \longrightarrow x/y$,

(iii) for every finite sequence T of elements of the types of it and for all types x, y of it such that $\langle y \rangle \frown T \longrightarrow x$ holds $T \longrightarrow y \setminus x$,

(iv) for all finite sequences T, X, Y of elements of the types of it and for all types x, y, z of it such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x/y \rangle \frown T \frown Y \longrightarrow z$,

(v) for all finite sequences T, X, Y of elements of the types of it and for all types x, y, z of it such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown T \frown \langle y \setminus x \rangle \frown Y \longrightarrow z$,

(vi) for all finite sequences X, Y of elements of the types of it and for all types x, y, z of it such that $X \frown \langle x \rangle \frown \langle y \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x \cdot y \rangle \frown Y \longrightarrow z$,

(vii) for all finite sequences X, Y of elements of the types of it and for all types x, y of it such that $X \longrightarrow x$ and $Y \longrightarrow y$ holds $X \frown Y \longrightarrow x \cdot y$.

In the sequel s will be a calculus of syntactic types and x, y, z will be types of s. The following propositions are true:

(12) $\langle x/y \rangle \frown \langle y \rangle \longrightarrow x$ and $\langle y \rangle \frown \langle y \setminus x \rangle \longrightarrow x$.

(13) $\langle x \rangle \longrightarrow y/(x \setminus y)$ and $\langle x \rangle \longrightarrow y/x \setminus y$.

(14) $\langle x/y \rangle \longrightarrow x/z/(y/z)$.

(15) $\langle y \setminus x \rangle \longrightarrow z \setminus y \setminus (z \setminus x)$.

(16) If $\langle x \rangle \longrightarrow y$, then $\langle x/z \rangle \longrightarrow y/z$ and $\langle z \setminus x \rangle \longrightarrow z \setminus y$.

(17) If $\langle x \rangle \longrightarrow y$, then $\langle z/y \rangle \longrightarrow z/x$ and $\langle y \setminus z \rangle \longrightarrow x \setminus z$.

(18) $\langle y/(y/x \setminus y) \rangle \longrightarrow y/x$.

(19) If $\langle x \rangle \longrightarrow y$, then $\varepsilon_{(\text{the types of } s)} \longrightarrow y/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus y$.

(20) $\varepsilon_{(\text{the types of } s)} \longrightarrow x/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus x$.

(21) $\varepsilon_{(\text{the types of } s)} \longrightarrow y/(x \setminus y)/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus (y/x \setminus y)$.

(22) $\varepsilon_{(\text{the types of } s)} \longrightarrow x/z/(y/z)/(x/y)$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow y \setminus x \setminus (z \setminus y \setminus (z \setminus x))$.

(23) If $\varepsilon_{(\text{the types of } s)} \longrightarrow x$, then $\varepsilon_{(\text{the types of } s)} \longrightarrow y/(y/x)$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus y \setminus y$.

(24) $\langle x/(y/y) \rangle \longrightarrow x$.

Let us consider s, x, y. The predicate $x \longleftrightarrow y$ is defined as follows:

(Def.20) $\langle x \rangle \longrightarrow y$ and $\langle y \rangle \longrightarrow x$.

Next we state several propositions:

(25) $x \longleftrightarrow x$.

(26) $x/y \longleftrightarrow x/(x/y \setminus x)$.

(27) $x/(z \cdot y) \longleftrightarrow x/y/z$.

(28) $\langle x \cdot (y/z) \rangle \longrightarrow (x \cdot y)/z$.

(29) $\langle x \rangle \longrightarrow (x \cdot y)/y$ and $\langle x \rangle \longrightarrow y \setminus y \cdot x$.

(30) $x \cdot y \cdot z \longleftrightarrow x \cdot (y \cdot z)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[4] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[5] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Joachim Lambek. The mathematics of sentence structure. *American Mathetical Monthly*, (65):154–170, 1958.
[14] Michał Muzalewski. Midpoint algebras. *Formalized Mathematics*, 1(**3**):483–488, 1990.
[15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[18] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 13, 1991