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On Projections in Projective Planes. Part II [1]

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Summary. We study in greater datail projectivities on Desarguesian projective planes. We are particularly interested in the situation when the composition of given two projectivities can be replaced by another two, with a given axis or centre of one of them.

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The articles [7], [9], [6], [8], [10], [11], [5], [4], [1], [2], and [3] provide the notation and terminology for this paper. In the sequel I_1 will denote a projective space defined in terms of incidence and z will denote an element of the points of I_1. Let us consider I_1, and let A, B, C be elements of the lines of I_1. We say that A, B, C are concurrent if and only if:

(Def.1) there exists an element o of the points of I_1 such that $o \mid A$ and $o \mid B$ and $o \mid C$.

Let us consider I_1, and let Z be an element of the lines of I_1. The functor chain(Z) yields a subset of the points of I_1 and is defined by:

(Def.2) $\mathrm{chain}(Z) = \{z : z \mid Z\}$.

We adopt the following rules: I_2 will denote an Desarguesian 2-dimensional projective space defined in terms of incidence, a, b, c, d, p, p'_1, q, o, o', o'', o'_1, r, s, x, y, o_1, o_2 will denote elements of the points of I_2, and O_1, O_2, O_3, A, B, C, O, Q, R, S will denote elements of the lines of I_2. Let us consider I_2. A partial function from the points of I_2 to the points of I_2 is said to be a projection of I_2 if:

(Def.3) there exist a, A, B such that $a \nmid A$ and $a \nmid B$ and it $= \pi_a(A \to B)$.

The following propositions are true:

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(1) If $A = B$ or $B = C$ or $C = A$, then A, B, C are concurrent.

(2) If A, B, C are concurrent, then A, C, B are concurrent and B, A, C are concurrent and B, C, A are concurrent and C, A, B are concurrent and C, B, A are concurrent.

(3) If $o \nmid A$ and $o \nmid B$ and $y \mid B$, then there exists x such that $x \mid A$ and $\pi_o(A \to B)(x) = y$.

(4) If $o \nmid A$ and $o \nmid B$, then $\operatorname{rng} \pi_o(A \to B) \subseteq$ the points of I_2.

(5) If $o \nmid A$ and $o \nmid B$, then $\operatorname{dom} \pi_o(A \to B) = \operatorname{chain}(A)$.

(6) If $o \nmid A$ and $o \nmid B$, then $\operatorname{rng} \pi_o(A \to B) = \operatorname{chain}(B)$.

(7) For an arbitrary x holds $x \in \operatorname{chain}(A)$ if and only if there exists a such that $x = a$ and $a \mid A$.

(8) If $o \nmid A$ and $o \nmid B$, then $\pi_o(A \to B)$ is one-to-one.

(9) If $o \nmid A$ and $o \nmid B$, then $\pi_o(A \to B)^{-1} = \pi_o(B \to A)$.

(10) For every projection f of I_2 holds f^{-1} is a projection of I_2.

(11) If $o \nmid A$, then $\pi_o(A \to A) = \operatorname{id}_{\operatorname{chain}(A)}$.

(12) $\operatorname{id}_{\operatorname{chain}(A)}$ is a projection of I_2.

(13) If $o \nmid A$ and $o \nmid B$ and $o \nmid C$, then $\pi_o(C \to B) \cdot \pi_o(A \to C) = \pi_o(A \to B)$.

(14) Suppose $o_1 \nmid O_1$ and $o_1 \nmid O_2$ and $o_2 \nmid O_2$ and $o_2 \nmid O_3$ and O_1, O_2, O_3 are concurrent and $O_1 \neq O_3$. Then there exists o such that $o \nmid O_1$ and $o \nmid O_3$ and $\pi_{o_2}(O_2 \to O_3) \cdot \pi_{o_1}(O_1 \to O_2) = \pi_o(O_1 \to O_3)$.

(15) Suppose that

(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) A, B, C are not concurrent,
(vi) $c \mid A$,
(vii) $c \mid C$,
(viii) $c \mid Q$,
(ix) $b \nmid Q$,
(x) $A \neq Q$,
(xi) $a \neq b$,
(xii) $b \neq q$,
(xiii) $a \mid O$,
(xiv) $b \mid O$,
(xv) B, C, O are not concurrent,
(xvi) $d \mid C$,
(xvii) $d \mid B$,
(xviii) $a \mid O_1$,
(xix) $d \mid O_1$,
(xx) $p \mid A$,
(xxi) $p \mid O_1$,

(xxii) $q \mid O$,
(xxiii) $q \mid O_2$,
(xxiv) $p \mid O_2$,
(xxv) $p'_1 \mid O_2$,
(xxvi) $d \mid O_3$,
(xxvii) $b \mid O_3$,
(xxviii) $p'_1 \mid O_3$,
(xxix) $p'_1 \mid Q$,
(xxx) $Q \neq C$,
(xxxi) $q \neq a$,
(xxxii) $q \nmid A$,
(xxxiii) $q \nmid Q$.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(16) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $b \neq q$,
(ix) $A \neq Q$,
(x) $c, o \mid A$,
(xi) $o, o'', d \mid B$,
(xii) $c, d, o' \mid C$,
(xiii) $a, b, d \mid O$,
(xiv) $c, o'_1 \mid Q$,
(xv) $a, o, o' \mid O_1$,
(xvi) $b, o', o'_1 \mid O_2$,
(xvii) $o, o'_1, q \mid O_3$,
(xviii) $q \mid O$.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(17) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) B, C, O are not concurrent,
(viii) $A \neq Q$,
(ix) $Q \neq C$,
(x) $a \neq b$,

(xi) $c, p \mid A$,
(xii) $d \mid B$,
(xiii) $c, d \mid C$,
(xiv) $a, b, q \mid O$,
(xv) $c, p'_1 \mid Q$,
(xvi) $a, d, p \mid O_1$,
(xvii) $q, p, p'_1 \mid O_2$,
(xviii) $b, d, p'_1 \mid O_3$.

Then $q \neq a$ and $q \neq b$ and $q \nmid A$ and $q \nmid Q$.

(18) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $A \neq Q$,
(ix) $c, o \mid A$,
(x) $o, o'', d \mid B$,
(xi) $c, d, o' \mid C$,
(xii) $a, b, d \mid O$,
(xiii) $c, o'_1 \mid Q$,
(xiv) $a, o, o' \mid O_1$,
(xv) $b, o', o'_1 \mid O_2$,
(xvi) $o, o'_1, q \mid O_3$,
(xvii) $q \mid O$.

Then $q \nmid A$ and $q \nmid Q$ and $b \neq q$.

(19) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $q \nmid A$,
(vi) A, B, C are not concurrent,
(vii) B, C, O are not concurrent,
(viii) $a \neq b$,
(ix) $b \neq q$,
(x) $q \neq a$,
(xi) $c, p \mid A$,
(xii) $d \mid B$,
(xiii) $c, d \mid C$,
(xiv) $a, b, q \mid O$,
(xv) $c, p'_1 \mid Q$,

(xvi) $a, d, p \mid O_1$,
(xvii) $q, p, p'_1 \mid O_2$,
(xviii) $b, d, p'_1 \mid O_3$.
Then $Q \neq A$ and $Q \neq C$ and $q \nmid Q$ and $b \nmid Q$.

(20) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $q \nmid A$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $b \neq q$,
(ix) $c, o \mid A$,
(x) $o, o'', d \mid B$,
(xi) $c, d, o' \mid C$,
(xii) $a, b, d \mid O$,
(xiii) $c, o'_1 \mid Q$,
(xiv) $a, o, o' \mid O_1$,
(xv) $b, o', o'_1 \mid O_2$,
(xvi) $o, o'_1, q \mid O_3$,
(xvii) $q \mid O$.
Then $b \nmid Q$ and $q \nmid Q$ and $A \neq Q$.

(21) Suppose that
(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) A, B, C are not concurrent,
(vi) A, C, Q are concurrent,
(vii) $b \nmid Q$,
(viii) $A \neq Q$,
(ix) $a \neq b$,
(x) $a \mid O$,
(xi) $b \mid O$.
Then there exists q such that $q \mid O$ and $q \nmid A$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(22) Suppose that
(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) A, B, C are not concurrent,
(vi) B, C, Q are concurrent,

(vii) $a \nmid Q$,
(viii) $B \neq Q$,
(ix) $a \neq b$,
(x) $a \mid O$,
(xi) $b \mid O$.

Then there exists q such that $q \mid O$ and $q \nmid B$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

(23) Suppose that
(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) $a \nmid B$,
(vi) $b \nmid A$,
(vii) $c \mid A$,
(viii) $c \mid C$,
(ix) $d \mid B$,
(x) $d \mid C$,
(xi) $a \mid S$,
(xii) $d \mid S$,
(xiii) $c \mid R$,
(xiv) $b \mid R$,
(xv) $s \mid A$,
(xvi) $s \mid S$,
(xvii) $r \mid B$,
(xviii) $r \mid R$,
(xix) $s \mid Q$,
(xx) $r \mid Q$,
(xxi) A, B, C are not concurrent.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_a(Q \to B) \cdot \pi_b(A \to Q)$.

(24) Suppose $a \nmid A$ and $b \nmid B$ and $a \nmid C$ and $b \nmid C$ and $a \neq b$ and $a \mid O$ and $b \mid O$ and $q \mid O$ and $q \nmid A$ and $q \neq b$ and A, B, C are not concurrent. Then there exists Q such that A, C, Q are concurrent and $b \nmid Q$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(25) Suppose $a \nmid A$ and $b \nmid B$ and $a \nmid C$ and $b \nmid C$ and $a \neq b$ and $a \mid O$ and $b \mid O$ and $q \mid O$ and $q \nmid B$ and $q \neq a$ and A, B, C are not concurrent. Then there exists Q such that B, C, Q are concurrent and $a \nmid Q$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[3] Eugeniusz Kusak and Wojciech Leończuk. Incidence projective space (a reduction theorem in a plane). *Formalized Mathematics*, 2(**2**):271–274, 1991.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.

[5] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[6] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[9] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[11] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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