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Oriented Metric-Affine Plane - Part I

Jarosław Zajkowski
Warsaw University
Białystok

Summary. We present (in Euclidean and Minkowskian geometry) definitions and some properties of the oriented orthogonality relation. Next we consider consistence of Euclidean space and consistence of Minkowskian space.

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The terminology and notation used in this paper have been introduced in the following articles: [1], [6], [7], [5], [3], [2], and [4]. We adopt the following rules: V will denote a real linear space, u, u_1, u_2, v, v_1, v_2, w, w_1, x, y will denote vectors of V, and n will denote a real number. Let us consider V, x, y. Let us assume that x, y span the space. Let us consider u. The functor $\rho^{\rm M}_{x,y}(u)$ yielding a vector of V is defined as follows:

(Def.1) $\rho^{\rm M}_{x,y}(u) = \pi^1_{x,y}(u) \cdot x + (-\pi^2_{x,y}(u)) \cdot y$.

The following propositions are true:

(1) If x, y span the space, then $\rho^{\rm M}_{x,y}(u + v) = \rho^{\rm M}_{x,y}(u) + \rho^{\rm M}_{x,y}(v)$.

(2) If x, y span the space, then $\rho^{\rm M}_{x,y}(n \cdot u) = n \cdot \rho^{\rm M}_{x,y}(u)$.

(3) If x, y span the space, then $\rho^{\rm M}_{x,y}(0_V) = 0_V$.

(4) If x, y span the space, then $\rho^{\rm M}_{x,y}(-u) = -\rho^{\rm M}_{x,y}(u)$.

(5) If x, y span the space, then $\rho^{\rm M}_{x,y}(u - v) = \rho^{\rm M}_{x,y}(u) - \rho^{\rm M}_{x,y}(v)$.

(6) If x, y span the space and $\rho^{\rm M}_{x,y}(u) = \rho^{\rm M}_{x,y}(v)$, then $u = v$.

(7) If x, y span the space, then $\rho^{\rm M}_{x,y}(\rho^{\rm M}_{x,y}(u)) = u$.

(8) If x, y span the space, then there exists v such that $u = \rho^{\rm M}_{x,y}(v)$.

Let us consider V, x, y. Let us assume that x, y span the space. Let us consider u. The functor $\rho^{\rm E}_{x,y}(u)$ yielding a vector of V is defined by:

(Def.2) $\rho^{\rm E}_{x,y}(u) = \pi^2_{x,y}(u) \cdot x + (-\pi^1_{x,y}(u)) \cdot y$.

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Next we state several propositions:

(9) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(-v) = -\rho^{\mathrm{E}}_{x,y}(v)$.

(10) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u+v) = \rho^{\mathrm{E}}_{x,y}(u) + \rho^{\mathrm{E}}_{x,y}(v)$.

(11) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u-v) = \rho^{\mathrm{E}}_{x,y}(u) - \rho^{\mathrm{E}}_{x,y}(v)$.

(12) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{E}}_{x,y}(u)$.

(13) If x, y span the space and $\rho^{\mathrm{E}}_{x,y}(u) = \rho^{\mathrm{E}}_{x,y}(v)$, then $u = v$.

(14) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(\rho^{\mathrm{E}}_{x,y}(u)) = -u$.

(15) If x, y span the space, then there exists v such that $\rho^{\mathrm{E}}_{x,y}(v) = u$.

We now define two new predicates. Let us consider V, x, y, u, v, u_1, v_1. Let us assume that x, y span the space. We say that the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y if and only if:

(Def.3) $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows u_1, v_1$.

We say that the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y if and only if:

(Def.4) $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows u_1, v_1$.

One can prove the following propositions:

(16) If x, y span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows \rho^{\mathrm{E}}_{x,y}(u_1), \rho^{\mathrm{E}}_{x,y}(v_1)$.

(17) If x, y span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows \rho^{\mathrm{M}}_{x,y}(u_1), \rho^{\mathrm{M}}_{x,y}(v_1)$.

(18) If x, y span the space, then if the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, v_1 and u_1, u are E-coherently orthogonal in the basis x, y.

(19) If x, y span the space, then if the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are M-coherently orthogonal in the basis x, y.

(20) If x, y span the space, then the segments u, u and v, w are E-coherently orthogonal in the basis x, y.

(21) If x, y span the space, then the segments u, u and v, w are M-coherently orthogonal in the basis x, y.

(22) If x, y span the space, then the segments u, v and w, w are E-coherently orthogonal in the basis x, y.

(23) If x, y span the space, then the segments u, v and w, w are M-coherently orthogonal in the basis x, y.

(24) If x, y span the space, then u, v, $\rho^{\mathrm{E}}_{x,y}(u)$ and $\rho^{\mathrm{E}}_{x,y}(v)$ are orthogonal w.r.t. x, y.

(25) If x, y span the space, then the segments u, v and $\rho^{\mathrm{E}}_{x,y}(u)$, $\rho^{\mathrm{E}}_{x,y}(v)$ are E-coherently orthogonal in the basis x, y.

(26) If x, y span the space, then the segments u, v and $\rho^{\mathrm{M}}_{x,y}(u)$, $\rho^{\mathrm{M}}_{x,y}(v)$ are M-coherently orthogonal in the basis x, y.

(27) If x, y span the space, then $u, v \upharpoonleft\!\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are E-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are E-coherently orthogonal in the basis x, y.

(28) If x, y span the space, then $u, v \upharpoonleft\!\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are M-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are M-coherently orthogonal in the basis x, y.

(29) If x, y span the space, then u, v, u_1 and v_1 are orthogonal w.r.t. x, y if and only if the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y or the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(30) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y, then $u = v$ or $u_1 = v_1$.

(31) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and v_1, u_1 are M-coherently orthogonal in the basis x, y, then $u = v$ or $u_1 = v_1$.

(32) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y, then the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y or the segments u, v and w, v_1 are E-coherently orthogonal in the basis x, y.

(33) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y, then the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y or the segments u, v and w, v_1 are M-coherently orthogonal in the basis x, y.

(34) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, u and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(35) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, u and v_1, u_1 are M-coherently orthogonal in the basis x, y.

(36) If x, y span the space and the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y and the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y, then the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y.

(37) If x, y span the space and the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y and the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y, then the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y.

(38) If x, y span the space, then for every u, v, w there exists u_1 such that

$w \neq u_1$ and the segments w, u_1 and u, v are E-coherently orthogonal in the basis x, y.

(39) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments w, u_1 and u, v are M-coherently orthogonal in the basis x, y.

(40) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are E-coherently orthogonal in the basis x, y.

(41) If x, y span the space, then for every u, v, w there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are M-coherently orthogonal in the basis x, y.

(42) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments w, w_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments w, w_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y, then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y.

(43) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments w, w_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments w, w_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y, then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y.

(44) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are E-coherently orthogonal in the basis x, y or the segments v, v_1 and u_1, u are E-coherently orthogonal in the basis x, y.

(45) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y, then the segments v, v_1 and u, u_1 are M-coherently orthogonal in the basis x, y or the segments v, v_1 and u_1, u are M-coherently orthogonal in the basis x, y.

(46) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y and the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y, then the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $w = w_1$.

Next we state several propositions:

(47) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y and the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y, then the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y or $v = v_1$ or

$w = w_1$.

(48) If x, y span the space and the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y and the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y, then the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

(49) If x, y span the space and the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y and the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y and the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y, then the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

(50) Suppose x, y span the space. Given v, w, u_1, v_1, w_1. Suppose the segments v, v_1 and w, u_1 are not E-coherently orthogonal in the basis x, y and the segments v, v_1 and u_1, w are not E-coherently orthogonal in the basis x, y and the segments u_1, w_1 and u_1, w are E-coherently orthogonal in the basis x, y. Then there exists u_2 such that the segments v, v_1 and v, u_2 are E-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are E-coherently orthogonal in the basis x, y but the segments u_1, w_1 and u_1, u_2 are E-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are E-coherently orthogonal in the basis x, y.

(51) If x, y span the space, then there exist u, v, w such that the segments u, v and u, w are E-coherently orthogonal in the basis x, y and for all v_1, w_1 such that the segments v_1, w_1 and u, v are E-coherently orthogonal in the basis x, y holds the segments v_1, w_1 and u, w are not E-coherently orthogonal in the basis x, y and the segments v_1, w_1 and w, u are not E-coherently orthogonal in the basis x, y or $v_1 = w_1$.

(52) Suppose x, y span the space. Given v, w, u_1, v_1, w_1. Suppose h the segments v, v_1 and w, u_1 are not M-coherently orthogonal in the basis x, y and h the segments v, v_1 and u_1, w are not M-coherently orthogonal in the basis x, y and the segments u_1, w_1 and u_1, w are M-coherently orthogonal in the basis x, y. Then there exists u_2 such that the segments v, v_1 and v, u_2 are M-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are M-coherently orthogonal in the basis x, y but the segments u_1, w_1 and u_1, u_2 are M-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are M-coherently orthogonal in the basis x, y.

(53) If x, y span the space, then there exist u, v, w such that the segments u, v and u, w are M-coherently orthogonal in the basis x, y and for all v_1, w_1 such that the segments v_1, w_1 and u, v are M-coherently orthogonal in the basis x, y holds h the segments v_1, w_1 and u, w are not M-coherently orthogonal in the basis x, y and h the segments v_1, w_1 and w, u are not M-coherently orthogonal in the basis x, y or $v_1 = w_1$.

In the sequel u_3, v_3 will be arbitrary. Let us consider V, x, y. Let us assume that x, y span the space. The Euclidean oriented orthogonality defined over V,x,y yielding a binary relation on [: the vectors of V, the vectors of V :] is defined as follows:

(Def.5) $\langle u_3, v_3 \rangle \in$the Euclidean oriented orthogonality defined over V,x,y if and only if there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2 \rangle$ and $v_3 = \langle v_1, v_2 \rangle$ and the segments u_1, u_2 and v_1, v_2 are E-coherently orthogonal in the basis x, y.

Let us consider V, x, y. Let us assume that x, y span the space. The Minkowskian oriented orthogonality defined over V,x,y yields a binary relation on [: the vectors of V, the vectors of V :] and is defined by:

(Def.6) $\langle u_3, v_3 \rangle \in$the Minkowskian oriented orthogonality defined over V,x,y if and only if there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2 \rangle$ and $v_3 = \langle v_1, v_2 \rangle$ and the segments u_1, u_2 and v_1, v_2 are M-coherently orthogonal in the basis x, y.

Let us consider V, x, y. Let us assume that x, y span the space. The functor CESpace(V, x, y) yields an affine structure and is defined by:

(Def.7) CESpace$(V, x, y) = \langle$ the vectors of V,the Euclidean oriented orthogonality defined over V,x,$y\rangle$.

Let us consider V, x, y. Let us assume that x, y span the space. The functor CMSpace(V, x, y) yielding an affine structure is defined by:

(Def.8) CMSpace$(V, x, y) = \langle$ the vectors of V,the Minkowskian oriented orthogonality defined over V,x,$y\rangle$.

Let A_1 be an affine structure, and let p, q, r, s be elements of the points of A_1. The predicate $p, q \top^{>} r, s$ is defined as follows:

(Def.9) $\langle \langle p, q \rangle, \langle r, s \rangle \rangle \in$ the congruence of A_1.

One can prove the following propositions:

(54) If x, y span the space, then for every u_3 holds u_3 is an element of the points of CESpace(V, x, y) if and only if u_3 is a vector of V.

(55) If x, y span the space, then for every u_3 holds u_3 is an element of the points of CMSpace(V, x, y) if and only if u_3 is a vector of V.

In the sequel p, q, r, s are elements of the points of CESpace(V, x, y). Next we state the proposition

(56) If x, y span the space and $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$, then $p, q \top^{>} r, s$ if and only if the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y.

In the sequel p, q, r, s will be elements of the points of CMSpace(V, x, y). We now state the proposition

(57) If x, y span the space and $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$, then $p, q \top^{>} r, s$ if and only if the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y.

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