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A Borsuk Theorem on Homotopy Types

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Summary. We present a Borsuk's theorem published first in [3] (compare also [4, pages 119–120]). It is slightly generalized, the assumption of metrizability is omitted. We introduce concepts needed for the formulation and the proof of theorems on upper semi-continuous decompositions, retracts, strong deformation retract. However, only those facts that are necessary in the proof have been proved.

MML Identifier: BORSUK_1.

The terminology and notation used here have been introduced in the following articles: [22], [7], [21], [2], [24], [23], [20], [12], [18], [14], [8], [13], [16], [25], [11], [10], [6], [5], [17], [1], [19], [9], and [15].

Preliminaries

We follow a convention: X, Y, X_1, X_2, Y_1, Y_2 will be sets, A will be a subset of X, and e, u will be arbitrary. The following propositions are true:

(1) If X meets Y_1 and $X \subseteq Y_2$, then X meets $Y_1 \cap Y_2$.

(2) If $e \in [\!:X_1, Y_1:\!]$ and $e \in [\!:X_2, Y_2:\!]$, then $e \in [\!:X_1 \cap X_2, Y_1 \cap Y_2:\!]$.

(3) $\mathrm{id}_X {}^\circ A = A$.

(4) $\mathrm{id}_X {}^{-1} A = A$.

(5) For every function F such that $X \subseteq F^{-1} X_1$ holds $F {}^\circ X \subseteq X_1$.

(6) $(X \longmapsto u) {}^\circ X_1 \subseteq \{u\}$.

(7) If $[\!:X_1, X_2:\!] \subseteq [\!:Y_1, Y_2:\!]$ and $[\!:X_1, X_2:\!] \neq \emptyset$, then $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$.

(8) If $\{e\}$ meets X, then $e \in X$.

The scheme *NonUniqExD* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

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provided the following requirement is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $u \in \mathcal{B}$ and $\mathcal{P}[e, u]$.

We now state several propositions:

(9) If $e \in 2^{[: X, Y :]}$, then $(^\circ \pi_1(X \times Y))(e) = \pi_1(X \times Y) ^\circ e$.

(10) If $e \in 2^{[: X, Y :]}$, then $(^\circ \pi_2(X \times Y))(e) = \pi_2(X \times Y) ^\circ e$.

(11) If $e \in [: X, Y :]$, then $e = \langle e_{\mathbf{1}}, e_{\mathbf{2}} \rangle$.

(12) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $\pi_1(X \times Y) ^\circ [: X_1, Y_1 :] = X_1$ and $\pi_2(X \times Y) ^\circ [: X_1, Y_1 :] = Y_1$.

(13) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $(^\circ \pi_1(X \times Y))([: X_1, Y_1 :]) = X_1$ and $(^\circ \pi_2(X \times Y))([: X_1, Y_1 :]) = Y_1$.

(14) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcup((^\circ \pi_1(X \times Y)) ^\circ H), \bigcap((^\circ \pi_2(X \times Y)) ^\circ H) :] \subseteq A$.

(15) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcap((^\circ \pi_1(X \times Y)) ^\circ H), \bigcup((^\circ \pi_2(X \times Y)) ^\circ H) :] \subseteq A$.

(16) For every set X and for every non-empty set Y and for every function f from X into Y and for every family H of subsets of X holds $\bigcup((^\circ f) ^\circ H) = f ^\circ \bigcup H$.

In the sequel X, Y, Z denote non-empty sets. One can prove the following propositions:

(17) For every family a of subsets of X holds $\bigcup \bigcup a = \bigcup \{\bigcup A : A \in a\}$, where A ranges over subsets of X.

(18) For every family D of subsets of X such that $\bigcup D = X$ for every subset A of D and for every subset B of X such that $B = \bigcup A$ holds $B^{\rm c} \subseteq \bigcup(A^{\rm c})$.

(19) For every function F from X into Y and for every function G from X into Z such that for all elements x, x' of X such that $F(x) = F(x')$ holds $G(x) = G(x')$ there exists a function H from Y into Z such that $H \cdot F = G$.

(20) For all X, Y, Z and for every element y of Y and for every function F from X into Y and for every function G from Y into Z holds $F^{-1} \{y\} \subseteq (G \cdot F)^{-1} \{G(y)\}$.

(21) For every function F from X into Y and for every element x of X and for every element z of Z holds $[: F, {\rm id}_Z :](\langle x, z \rangle) = \langle F(x), z \rangle$.

(22) For every function F from X into Y and for every subset A of X holds ${\rm id}_X ^\circ A = A$.

(23) For every function F from X into Y and for every subset A of X and for every subset B of Z holds $[: F, {\rm id}_Z :] ^\circ [: A, B :] = [: F ^\circ A, B :]$.

(24) For every function F from X into Y and for every element y of Y and for every element z of Z holds $[: F, \mathrm{id}_Z :]^{-1} \{\langle y, z \rangle\} = [: F^{-1} \{y\}, \{z\} :]$.

Let B, A be non-empty sets, and let x be an element of B. Then $A \longmapsto x$ is a function from A into B.

Let Y be a non-empty set, and let y be an element of Y. Then $\{y\}$ is a subset of Y.

Partitions

One can prove the following four propositions:

(25) For every partition D of X and for every subset A of D holds $\bigcup A$ is a subset of X.

(26) For every partition D of X and for all subsets A, B of D holds $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(27) For every partition D of X and for every subset A of D and for every subset B of X such that $B = \bigcup A$ holds $B^{\mathrm{c}} = \bigcup(A^{\mathrm{c}})$.

(28) For every equivalence relation E of X holds Classes E is non-empty.

Let us consider X, and let D be a non-empty partition of X. The projection onto D yielding a function from X into D is defined as follows:

(Def.1) for every element p of X holds $p \in$ (the projection onto D)(p).

Next we state several propositions:

(29) For every non-empty partition D of X and for every element p of X and for every element A of D such that $p \in A$ holds $A =$ (the projection onto D)(p).

(30) For every non-empty partition D of X and for every element p of D holds $p =$ (the projection onto D)$^{-1} \{p\}$.

(31) For every non-empty partition D of X and for every subset A of D holds (the projection onto D)$^{-1} A = \bigcup A$.

(32) For every non-empty partition D of X and for every element W of D there exists an element W' of X such that (the projection onto D)(W') $= W$.

(33) For every non-empty partition D of X and for every subset W of X such that for every subset B of X such that $B \in D$ and B meets W holds $B \subseteq W$ holds $W =$ (the projection onto D)$^{-1}$(the projection onto D)$^{\circ} W$.

Topological Preliminaries

In the sequel X, Y denote topological spaces. We now state two propositions:

(34) $\Omega_X \neq \emptyset_X$.

(35) For every subspace Y of X holds the carrier of $Y \subseteq$ the carrier of X.

Let X, Y be topological spaces, and let F be a function from the carrier of X into the carrier of Y. Let us note that one can characterize the predicate F

is continuous by the following (equivalent) condition:

(Def.2) for every point W of X and for every neighborhood G of $F(W)$ there exists a neighborhood H of W such that $F^{\circ} H \subseteq G$.

The following proposition is true

(36) For every point y of Y holds (the carrier of X) $\longmapsto y$ is continuous.

Let us consider X, Y. A map from X into Y is called a continuous map from X into Y if:

(Def.3) it is continuous.

Let X, Y, Z be topological spaces, and let F be a continuous map from X into Y, and let G be a continuous map from Y into Z. Then $G \cdot F$ is a continuous map from X into Z.

We now state two propositions:

(37) For every continuous map A from X into Y and for every subset G of Y holds $A^{-1} \operatorname{Int} G \subseteq \operatorname{Int}(A^{-1} G)$.

(38) For every point W of Y and for every continuous map A from X into Y and for every neighborhood G of W holds $A^{-1} G$ is a neighborhood of $A^{-1} \{W\}$.

Let X, Y be topological spaces, and let W be a point of Y, and let A be a continuous map from X into Y, and let G be a neighborhood of W. Then $A^{-1} G$ is a neighborhood of $A^{-1} \{W\}$.

One can prove the following propositions:

(39) For every X and for all subsets A, B of the carrier of X and for every neighborhood U_1 of B such that $A \subseteq B$ holds U_1 is a neighborhood of A.

(40) For every subset A of X and for every point x of X holds A is a neighborhood of x if and only if A is a neighborhood of $\{x\}$.

(41) For every point x of X holds $\{x\}$ is compact.

(42) For every subspace Y of X and for every subset A of X and for every subset B of Y such that $A = B$ holds A is compact if and only if B is compact.

Cartesian Products of Topological Spaces

Let us consider X, Y. The functor $[\!: X, Y :\!]$ yielding a topological space is defined by:

(Def.4) the carrier of $[\!: X, Y :\!] = [\!:$ the carrier of $X,$ the carrier of $Y :\!]$ and the topology of $[\!: X, Y :\!] = \{\bigcup A : A \subseteq \{[\!: X_1, Y_1 :\!] : X_1 \in$ the topology of $X \wedge Y_1 \in$ the topology of $Y\}\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

Next we state three propositions:

(43) The carrier of $[\!: X, Y :\!] = [\!:$ the carrier of $X,$ the carrier of $Y :\!]$.

(44) The topology of $[: X, Y :] = \{\bigcup A : A \subseteq \{[: X_1, Y_1 :] : X_1 \in$ the topology of $X \wedge Y_1 \in$ the topology of $Y\}\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

(45) For every subset B of $[: X, Y :]$ holds B is open if and only if there exists a family A of subsets of the carrier of $[: X, Y :]$ such that $B = \bigcup A$ and for every e such that $e \in A$ there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ and X_1 is open and Y_1 is open.

Let X, Y be topological spaces, and let A be a subset of X, and let B be a subset of Y. Then $[: A, B :]$ is a subset of $[: X, Y :]$.

Let X, Y be topological spaces, and let x be a point of X, and let y be a point of Y. Then $\langle x, y\rangle$ is a point of $[: X, Y :]$.

Next we state four propositions:

(46) For every subset V of X and for every subset W of Y such that V is open and W is open holds $[: V, W :]$ is open.

(47) For every subset V of X and for every subset W of Y holds $\mathrm{Int}[: V, W :] = [: \mathrm{Int}\, V, \mathrm{Int}\, W :]$.

(48) For every point x of X and for every point y of Y and for every neighborhood V of x and for every neighborhood W of y holds $[: V, W :]$ is a neighborhood of $\langle x, y\rangle$.

(49) For every subset A of X and for every subset B of Y and for every neighborhood V of A and for every neighborhood W of B holds $[: V, W :]$ is a neighborhood of $[: A, B :]$.

Let X, Y be topological spaces, and let x be a point of X, and let y be a point of Y, and let V be a neighborhood of x, and let W be a neighborhood of y. Then $[: V, W :]$ is a neighborhood of $\langle x, y\rangle$.

Next we state the proposition

(50) For every point X_3 of $[: X, Y :]$ there exists a point W of X and there exists a point T of Y such that $X_3 = \langle W, T\rangle$.

Let X, Y be topological spaces, and let A be a subset of X, and let t be a point of Y, and let V be a neighborhood of A, and let W be a neighborhood of t. Then $[: V, W :]$ is a neighborhood of $[: A, \{t\} :]$.

Let us consider X, Y, and let A be a subset of $[: X, Y :]$. The functor BaseAppr(A) yields a family of subsets of $[: X, Y :]$ and is defined by:

(Def.5) $\mathrm{BaseAppr}(A) = \{[: X_1, Y_1 :] : [: X_1, Y_1 :] \subseteq A \wedge X_1$ is open$\wedge Y_1$ is open$\}$, where X_1 ranges over subsets of X, and Y_1 ranges over subsets of Y.

We now state several propositions:

(51) For every subset A of $[: X, Y :]$ holds BaseAppr(A) is open.

(52) For all subsets A, B of $[: X, Y :]$ such that $A \subseteq B$ holds $\mathrm{BaseAppr}(A) \subseteq \mathrm{BaseAppr}(B)$.

(53) For every subset A of $[: X, Y :]$ holds $\bigcup \mathrm{BaseAppr}(A) \subseteq A$.

(54) For every subset A of $[: X, Y :]$ such that A is open holds $A = \bigcup \mathrm{BaseAppr}(A)$.

(55) For every subset A of $[: X, Y :]$ holds $\operatorname{Int} A = \bigcup \mathrm{BaseAppr}(A)$.

We now define two new functors. Let us consider X, Y. The functor $\pi_1(X,Y)$ yielding a function from $2^{\text{the carrier of } [: X, Y :]}$ into $2^{\text{the carrier of } X}$ is defined by:

(Def.6) $\pi_1(X,Y) = {}^\circ\pi_1($ (the carrier of $X) \times$ the carrier of $Y)$.

The functor $\pi_2(X,Y)$ yields a function from $2^{\text{the carrier of } [: X, Y :]}$ into $2^{\text{the carrier of } Y}$ and is defined as follows:

(Def.7) $\pi_2(X,Y) = {}^\circ\pi_2($ (the carrier of $X) \times$ the carrier of $Y)$.

We now state a number of propositions:

(56) Let A be a subset of $[: X, Y :]$. Then for every family H of subsets of $[: X, Y :]$ such that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [: X_1, Y_1 :]$ holds $[: \bigcup(\pi_1(X,Y) \circ H), \bigcap(\pi_2(X,Y) \circ H) :] \subseteq A$.

(57) For every family H of subsets of $[: X, Y :]$ and for every set C such that $C \in \pi_1(X,Y) \circ H$ there exists a subset D of $[: X, Y :]$ such that $D \in H$ and $C = \pi_1($ (the carrier of $X) \times$ the carrier of $Y) \circ D$.

(58) For every family H of subsets of $[: X, Y :]$ and for every set C such that $C \in \pi_2(X,Y) \circ H$ there exists a subset D of $[: X, Y :]$ such that $D \in H$ and $C = \pi_2($ (the carrier of $X) \times$ the carrier of $Y) \circ D$.

(59) For every subset D of $[: X, Y :]$ such that D is open for every subset X_1 of X and for every subset Y_1 of Y holds if $X_1 = \pi_1($ (the carrier of $X) \times$ the carrier of $Y) \circ D$, then X_1 is open but if $Y_1 = \pi_2($ (the carrier of $X) \times$ the carrier of $Y) \circ D$, then Y_1 is open.

(60) For every family H of subsets of $[: X, Y :]$ such that H is open holds $\pi_1(X,Y) \circ H$ is open and $\pi_2(X,Y) \circ H$ is open.

(61) For every family H of subsets of $[: X, Y :]$ such that $\pi_1(X,Y) \circ H = \emptyset$ or $\pi_2(X,Y) \circ H = \emptyset$ holds $H = \emptyset$.

(62) For every family H of subsets of $[: X, Y :]$ and for every subset X_1 of X and for every subset Y_1 of Y such that H is a cover of $[: X_1, Y_1 :]$ holds if $Y_1 \neq \emptyset$, then $\pi_1(X,Y) \circ H$ is a cover of X_1 but if $X_1 \neq \emptyset$, then $\pi_2(X,Y) \circ H$ is a cover of Y_1.

(63) For every family H of subsets of X and for every subset Y of X such that H is a cover of Y there exists a family F of subsets of X such that $F \subseteq H$ and F is a cover of Y and for every set C such that $C \in F$ holds $C \cap Y \neq \emptyset$.

(64) For every family F of subsets of X and for every family H of subsets of $[: X, Y :]$ such that F is finite and $F \subseteq \pi_1(X,Y) \circ H$ there exists a family G of subsets of $[: X, Y :]$ such that $G \subseteq H$ and G is finite and $F = \pi_1(X,Y) \circ G$.

(65) For every subset X_1 of X and for every subset Y_1 of Y such that $[: X_1, Y_1 :] \neq \emptyset$ holds $\pi_1(X,Y)([: X_1, Y_1 :]) = X_1$ and $\pi_2(X,Y)([: X_1, Y_1 :]) = Y_1$.

(66) $\pi_1(X,Y)(\emptyset) = \emptyset$ and $\pi_2(X,Y)(\emptyset) = \emptyset$.

(67) For every point t of Y and for every subset A of the carrier of X such that A is compact for every neighborhood G of $[: A, \{t\} :]$ there exists a neighborhood V of A and there exists a neighborhood W of t such that $[: V, W :] \subseteq G$.

Partitions of Topological Spaces

Let us consider X. The trivial decomposition of X yielding a non-empty partition of the carrier of X is defined by:

(Def.8) the trivial decomposition of $X = \mathrm{Classes}(\triangle_{\text{the carrier of } X})$.

We now state the proposition

(68) For every subset A of X such that $A \in$ the trivial decomposition of X there exists a point x of X such that $A = \{x\}$.

Let X be a topological space, and let D be a non-empty partition of the carrier of X. The decomposition space of D yielding a topological space is defined as follows:

(Def.9) the carrier of the decomposition space of $D = D$ and the topology of the decomposition space of $D = \{A : \bigcup A \in$ the topology of $X\}$, where A ranges over subsets of D.

One can prove the following proposition

(69) For every non-empty partition D of the carrier of X and for every subset A of D holds $\bigcup A \in$ the topology of X if and only if $A \in$ the topology of the decomposition space of D.

Let X be a topological space, and let D be a non-empty partition of the carrier of X. The projection onto D yielding a continuous map from X into the decomposition space of D is defined as follows:

(Def.10) the projection onto $D =$ the projection onto D.

We now state three propositions:

(70) For every non-empty partition D of the carrier of X and for every point W of X holds $W \in$ (the projection onto D)(W).

(71) For every non-empty partition D of the carrier of X and for every point W of the decomposition space of D there exists a point W' of X such that (the projection onto D)(W') $= W$.

(72) For every non-empty partition D of the carrier of X holds rng(the projection onto D) = the carrier of the decomposition space of D.

Let X_4 be a topological space, and let X be a subspace of X_4, and let D be a non-empty partition of the carrier of X. The trivial extension of D yields a non-empty partition of the carrier of X_4 and is defined as follows:

(Def.11) the trivial extension of $D = D \cup \{\{p\} : p \notin$ the carrier of $X\}$, where p ranges over points of X_4.

The following propositions are true:

(73) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X holds $D \subseteq$ the trivial extension of D.

(74) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every subset A of X_4 such that $A \in$ the trivial extension of D holds $A \in D$ or there exists a point x of X_4 such that $x \notin \Omega_X$ and $A = \{x\}$.

(75) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point x of X_4 such that $x \notin$ the carrier of X holds $\{x\} \in$ the trivial extension of D.

(76) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point W of X_4 such that $W \in$ the carrier of X holds (the projection onto the trivial extension of $D)(W) =$ (the projection onto $D)(W)$.

(77) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point W of X_4 such that $W \notin$ the carrier of X holds (the projection onto the trivial extension of $D)(W) = \{W\}$.

(78) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for all points W, W' of X_4 such that $W \notin$ the carrier of X and (the projection onto the trivial extension of $D)(W) =$ (the projection onto the trivial extension of $D)(W')$ holds $W = W'$.

(79) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every point e of X_4 such that (the projection onto the trivial extension of $D)(e) \in$ the carrier of the decomposition space of D holds $e \in$ the carrier of X.

(80) For every topological space X_4 and for every subspace X of X_4 and for every non-empty partition D of the carrier of X and for every e such that $e \in$ the carrier of X holds (the projection onto the trivial extension of $D)(e) \in$ the carrier of the decomposition space of D.

Upper Semicontinuous Decompositions

Let X be a topological space. A non-empty partition of the carrier of X is said to be an upper semi-continuous decomposition of X if:

(Def.12) for every subset A of X such that $A \in$ it for every neighborhood V of A there exists a subset W of X such that W is open and $A \subseteq W$ and $W \subseteq V$ and for every subset B of X such that $B \in$ it and B meets W holds $B \subseteq W$.

We now state two propositions:

(81) For every upper semi-continuous decomposition D of X and for every point t of the decomposition space of D and for every neighborhood G

of (the projection onto D) $^{-1}$ $\{t\}$ holds (the projection onto D) $\circ$ G is a neighborhood of t.

(82) The trivial decomposition of X is an upper semi-continuous decomposition of X.

Let us consider X. A subspace of X is called a closed subspace of X if:

(Def.13) for every subset A of X such that $A =$ the carrier of it holds A is closed.

Let X_4 be a topological space, and let X be a closed subspace of X_4, and let D be an upper semi-continuous decomposition of X. Then the trivial extension of D is an upper semi-continuous decomposition of X_4.

Let X be a topological space. An upper semi-continuous decomposition of X is called an upper semi-continuous decomposition into compacta of X if:

(Def.14) for every subset A of X such that $A \in$ it holds A is compact.

Let X_4 be a topological space, and let X be a closed subspace of X_4, and let D be an upper semi-continuous decomposition into compacta of X. Then the trivial extension of D is an upper semi-continuous decomposition into compacta of X_4.

Let X be a topological space, and let Y be a closed subspace of X, and let D be an upper semi-continuous decomposition into compacta of Y. Then the decomposition space of D is a closed subspace of the decomposition space of the trivial extension of D.

Borsuk's Theorems on the Decomposition of Retracts

The topological space $\mathbb{I}$ is defined by:

(Def.15) for every subset P of (the metric space of real numbers)$_{\rm top}$ such that $P = [0, 1]$ holds $\mathbb{I} =$ (the metric space of real numbers)$_{\rm top} \restriction P$.

Next we state the proposition

(83) The carrier of $\mathbb{I} = [0, 1]$.

We now define two new functors. The point $0_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def.16) $0_{\mathbb{I}} = 0$.

The point $1_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def.17) $1_{\mathbb{I}} = 1$.

Let A be a topological space, and let B be a subspace of A, and let F be a continuous map from A into B. We say that F is a retraction if and only if:

(Def.18) for every point W of A such that $W \in$ the carrier of B holds $F(W) = W$.

We now define two new predicates. Let X be a topological space, and let Y be a subspace of X. We say that Y is a retract of X if and only if:

(Def.19) there exists a continuous map F from X into Y such that F is a retraction.

We say that Y is a strong deformation retract of X if and only if:

(Def.20) there exists a continuous map H from $[: X, \mathbb{I} :]$ into X such that for every point A of X holds $H(\langle A, 0_{\mathbb{I}} \rangle) = A$ and $H(\langle A, 1_{\mathbb{I}} \rangle) \in$ the carrier of Y but if $A \in$ the carrier of Y, then for every point T of $\mathbb{I}$ holds $H(\langle A, T \rangle) = A$.

We now state two propositions:

(84) For every topological space X_4 and for every closed subspace X of X_4 and for every upper semi-continuous decomposition D into compacta of X such that X is a retract of X_4 holds the decomposition space of D is a retract of the decomposition space of the trivial extension of D.

(85) For every topological space X_4 and for every closed subspace X of X_4 and for every upper semi-continuous decomposition D into compacta of X such that X is a strong deformation retract of X_4 holds the decomposition space of D is a strong deformation retract of the decomposition space of the trivial extension of D.

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