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Atlas of Midpoint Algebra

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Summary. This article is a continuation of [4]. We have established a one-to-one correspondence between midpoint algebras and groups with the operator $\frac{1}{2}$. In general we shall say that a given midpoint algebra M and a group V are w-assotiated iff w is an atlas from M to V. At the beginning of the paper a few facts which rather belong to [3], [5] are proved.

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The terminology and notation used here have been introduced in the following articles: [2], [1], [3], [4], and [5]. In the sequel G is a group structure and x is an element of G. Let us consider G, x. The functor $2x$ yielding an element of G is defined by:

(Def.1) $2x = x + x$.

In the sequel M is a midpoint algebra structure. Let us consider M. A point of M is an element of the points of M.

In the sequel p, q, r will be points of M and w will be a function from [: the points of M, the points of M :] into the carrier of G. Let us consider M, G, w. We say that M, G are associated w.r.t. w if and only if:

(Def.2) $p \oplus q = r$ if and only if $w(p, r) = w(r, q)$.

The following proposition is true

(1) If M, G are associated w.r.t. w, then $p \oplus p = p$.

We follow the rules: S will be a non-empty set, a, b, b', c, c', d will be elements of S, and w will be a function from [: S, S :] into the carrier of G. Let us consider S, G, w. We say that w is an atlas of S, G if and only if:

(Def.3) for every a, x there exists b such that $w(a, b) = x$ and for all a, b, c such that $w(a, b) = w(a, c)$ holds $b = c$ and for all a, b, c holds $w(a, b) + w(b, c) = w(a, c)$.

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Let us consider S, G, w, a, x. Let us assume that w is an atlas of S, G. The functor $(a,x).w$ yielding an element of S is defined by:

(Def.4) $w(a,\ (a,x).w) = x$.

In the sequel G denotes a group, x, y denote elements of G, and w denotes a function from $[\!:S,\ S:\!]$ into the carrier of G. One can prove the following propositions:

(2) $2(0_G) = 0_G$.

(3) If $x + y = x$, then $y = 0_G$.

(4) If w is an atlas of S, G, then $w(a,\ a) = 0_G$.

(5) If w is an atlas of S, G and $w(a,\ b) = 0_G$, then $a = b$.

(6) If w is an atlas of S, G, then $w(a,\ b) = -w(b,\ a)$.

(7) If w is an atlas of S, G and $w(a,\ b) = w(c,\ d)$, then $w(b,\ a) = w(d,\ c)$.

(8) If w is an atlas of S, G, then for every b, x there exists a such that $w(a,\ b) = x$.

(9) If w is an atlas of S, G and $w(b,\ a) = w(c,\ a)$, then $b = c$.

(10) For every function w from $[\!:$ the points of M, the points of $M\,:\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w holds $p \oplus q = q \oplus p$.

(11) For every function w from $[\!:$ the points of M, the points of $M\,:\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w there exists r such that $r \oplus p = q$.

We adopt the following rules: G will denote an Abelian group and x, y, z, t will denote elements of G. The following propositions are true:

(12) $-(x + y) = -x + -y$.

(13) $x + y + (z + t) = x + z + (y + t)$.

(14) $2(x + y) = 2x + 2y$.

(15) $2(-x) = -2x$.

(16) For every function w from $[\!:$ the points of M, the points of $M\,:\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w for all points a, b, c, d of M holds $a \oplus b = c \oplus d$ if and only if $w(a,\ d) = w(c,\ b)$.

In the sequel w denotes a function from $[\!:S,\ S:\!]$ into the carrier of G. Next we state the proposition

(17) If w is an atlas of S, G, then for all a, b, b', c, c' such that $w(a,\ b) = w(b,\ c)$ and $w(a,\ b') = w(b',\ c')$ holds $w(c,\ c') = 2w(b,\ b')$.

We follow the rules: M denotes a midpoint algebra and p, q, r, s denote points of M. Let us consider M. Then vectgroup M is an Abelian group.

The following proposition is true

(18) For an arbitrary a holds a is an element of vectgroup M if and only if a is a vector of M and $0_{\mathrm{vectgroup}\,M} = \mathrm{I}_M$ and for all elements a, b of

vectgroup M and for all vectors x, y of M such that $a = x$ and $b = y$ holds $a + b = x + y$.

An Abelian group is called a group with the operator $\frac{1}{2}$ if:

(Def.5) for every element a of it there exists an element x of it such that $2x = a$ and for every element a of it such that $2a = 0_{\rm it}$ holds $a = 0_{\rm it}$.

In the sequel G is a group with the operator $\frac{1}{2}$ and x, y are elements of G. One can prove the following two propositions:

(19) If $x = -x$, then $x = 0_G$.

(20) If $2x = 2y$, then $x = y$.

Let us consider G, x. The functor $\frac{1}{2}x$ yielding an element of G is defined as follows:

(Def.6) $2\frac{1}{2}x = x$.

The following three propositions are true:

(21) $\frac{1}{2}(0_G) = 0_G$ and $\frac{1}{2}(x + y) = \frac{1}{2}x + \frac{1}{2}y$ but if $\frac{1}{2}x = \frac{1}{2}y$, then $x = y$ and $\frac{1}{2}2x = x$.

(22) For every M being a midpoint algebra structure and for every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w for all points a, b, c, d of M holds $a \oplus b \oplus (c \oplus d) = a \oplus c \oplus (b \oplus d)$.

(23) For every M being a midpoint algebra structure and for every function w from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G such that w is an atlas of the points of M, G and M, G are associated w.r.t. w holds M is a midpoint algebra.

Let us consider M. Then vectgroup M is a group with the operator $\frac{1}{2}$.

Let us consider M, p, q. The functor q^p yields an element of vectgroup M and is defined as follows:

(Def.7) $q^p = \overrightarrow{[p, q]}$.

Let us consider M. The functor vect M yields a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of vectgroup M and is defined by:

(Def.8) $(\mathrm{vect}\, M)(p,\, q) = \overrightarrow{[p, q]}$.

We now state four propositions:

(24) vect M is an atlas of the points of M, vectgroup M.

(25) $\overrightarrow{[p, q]} = \overrightarrow{[r, s]}$ if and only if $p \oplus s = q \oplus r$.

(26) $p \oplus q = r$ if and only if $\overrightarrow{[p, r]} = \overrightarrow{[r, q]}$.

(27) M, vectgroup M are associated w.r.t. vect M.

In the sequel w will denote a function from $[\!: S,\, S :\!]$ into the carrier of G. Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor ${}^{@}w$ yielding a binary operation on S is defined as follows:

(Def.9) $w(a,\, ({}^{@}w)(a,\, b)) = w(({}^{@}w)(a,\, b),\, b)$.

We now state the proposition

(28) If w is an atlas of S, G, then for all a, b, c holds $({}^{@}w)(a, b) = c$ if and only if $w(a, c) = w(c, b)$.

In the sequel a, b, c are points of $\langle S, {}^{@}w\rangle$. We now state two propositions:

(29) $({}^{@}w)(a, b) = a \oplus b$.

(30) $a \oplus b = c$ if and only if $({}^{@}w)(a, b) = c$.

Let us consider S, G, w. The functor Atlas w yielding a function from $[\!:$ the points of $\langle S, {}^{@}w\rangle$, the points of $\langle S, {}^{@}w\rangle :\!]$ into the carrier of G is defined as follows:

(Def.10) Atlas $w = w$.

Next we state two propositions:

(31) If w is an atlas of S, G, then Atlas w is an atlas of the points of $\langle S, {}^{@}w\rangle$, G.

(32) If w is an atlas of S, G, then $\langle S, {}^{@}w\rangle$, G are associated w.r.t. Atlas w.

Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor MidSp(w) yielding a midpoint algebra is defined by:

(Def.11) MidSp$(w) = \langle S, {}^{@}w\rangle$.

We follow the rules: M is a midpoint algebra structure, w is a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of G, and a, b, b_1, b_2, c are points of M. The following proposition is true

(33) M is a midpoint algebra if and only if there exists G and there exists w such that w is an atlas of the points of M, G and M, G are associated w.r.t. w.

Let us consider M. We consider atlas structures over M which are systems
⟨an algebra, a function⟩,
where the algebra is a group with the operator $\frac{1}{2}$ and the function is a function from $[\!:$ the points of M, the points of $M :\!]$ into the carrier of the algebra.

Let M be a midpoint algebra. An atlas structure over M is said to be an atlas of M if:

(Def.12) M, the algebra of it are associated w.r.t. the function of it and the function of it is an atlas of the points of M, the algebra of it.

Let M be a midpoint algebra, and let W be an atlas of M. A vector of W is an element of the algebra of W.

Let M be a midpoint algebra, and let W be an atlas of M, and let a, b be points of M. The functor $W(a, b)$ yields an element of the algebra of W and is defined as follows:

(Def.13) $W(a, b) =$ (the function of W)(a, b).

Let M be a midpoint algebra, and let W be an atlas of M, and let a be a point of M, and let x be a vector of W. The functor $(a, x).W$ yielding a point of M is defined as follows:

(Def.14) $(a, x).W = (a, x).$(the function of W).

Let M be a midpoint algebra, and let W be an atlas of M. The functor 0_W yielding a vector of W is defined as follows:

(Def.15) $0_W = 0_{\text{the algebra of } W}$.

We now state two propositions:

(34) If w is an atlas of the points of M, G and M, G are associated w.r.t. w, then $a \oplus c = b_1 \oplus b_2$ if and only if $w(a, c) = w(a, b_1) + w(a, b_2)$.

(35) If w is an atlas of the points of M, G and M, G are associated w.r.t. w, then $a \oplus c = b$ if and only if $w(a, c) = 2w(a, b)$.

For simplicity we adopt the following convention: M will be a midpoint algebra, W will be an atlas of M, a, b, b_1, b_2, c, d will be points of M, and x will be a vector of W. One can prove the following propositions:

(36) $a \oplus c = b_1 \oplus b_2$ if and only if $W(a, c) = W(a, b_1) + W(a, b_2)$.

(37) $a \oplus c = b$ if and only if $W(a, c) = 2W(a, b)$.

(38) For every a, x there exists b such that $W(a, b) = x$ and for all a, b, c such that $W(a, b) = W(a, c)$ holds $b = c$ and for all a, b, c holds $W(a, b) + W(b, c) = W(a, c)$.

(39) (i) $W(a, a) = 0_W$,

(ii) if $W(a, b) = 0_W$, then $a = b$,

(iii) $W(a, b) = -W(b, a)$,

(iv) if $W(a, b) = W(c, d)$, then $W(b, a) = W(d, c)$,

(v) for every b, x there exists a such that $W(a, b) = x$,

(vi) if $W(b, a) = W(c, a)$, then $b = c$,

(vii) $a \oplus b = c$ if and only if $W(a, c) = W(c, b)$,

(viii) $a \oplus b = c \oplus d$ if and only if $W(a, d) = W(c, b)$,

(ix) $W(a, b) = x$ if and only if $(a, x).W = b$.

(40) $(a, 0_W).W = a$.

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