FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Modal Propositional Logic

Alicia de la Cruz
Universidad Politecnica de Madrid

MML Identifier: MODAL_1.

The terminology and notation used here are introduced in the following papers: [15], [11], [2], [14], [16], [13], [7], [5], [6], [8], [10], [12], [1], [9], [3], [4], and [17]. For simplicity we follow a convention: x, y will be arbitrary, n, m, k will denote natural numbers, t_1 will denote a tree decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set $:\!]$, w, s, t will denote finite sequences of elements of $\mathbb{N}$, X will denote a set, and D will denote a non-empty set. Next we state the proposition

(1) If X is finite, then $\operatorname{card} X = 2$ if and only if there exist x, y such that $X = \{x, y\}$ and $x \neq y$.

Let Z be a tree. The root of Z yields an element of Z and is defined as follows:

(Def.1) the root of $Z = \varepsilon$.

Let us consider D, and let T be a tree decorated by D. The root of T yields an element of D and is defined by:

(Def.2) the root of $T = T$(the root of $\operatorname{dom} T$).

Next we state a number of propositions:

(2) $\langle n \rangle = \langle m \rangle$ if and only if $n = m$.

(3) If $n \neq m$, then $\langle n \rangle$ and $\langle m \rangle \frown s$ are not comparable.

(4) For every s such that $s \neq \varepsilon$ there exist w, n such that $s = \langle n \rangle \frown w$.

(5) If $n \neq m$, then $\langle n \rangle \not\prec \langle m \rangle \frown s$.

(6) If $n \neq m$, then $\langle n \rangle \not\preceq \langle m \rangle \frown s$.

(7) $\langle n \rangle \not\prec \langle m \rangle$.

(8) If $w \neq \varepsilon$, then $s \prec s \frown w$.

(9) The elementary tree of $1 = \{\varepsilon, \langle 0 \rangle\}$.

(10) The elementary tree of $2 = \{\varepsilon, \langle 0 \rangle, \langle 1 \rangle\}$.

(11) For every tree Z and for all n, m such that $n \leq m$ and $\langle m \rangle \in Z$ holds $\langle n \rangle \in Z$.

ISSN 0777–4028

(12) If $w \frown t \prec w \frown s$, then $t \prec s$.

(13) $t_1 \in \mathbb{N}^* \dot{\rightarrow} [\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set $:\!]$.

(14) For all trees Z, Z_1 and for every element z of Z holds $z \in Z(z/Z_1)$.

(15) For all trees Z, Z_1, Z_2 and for every element z of Z such that $Z(z/Z_1) = Z(z/Z_2)$ holds $Z_1 = Z_2$.

(16) For all trees Z, Z_1, Z_2 decorated by D and for every element z of $\mathrm{dom}\, Z$ such that $Z(z/Z_1) = Z(z/Z_2)$ holds $Z_1 = Z_2$.

(17) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_1 such that $v = w$ and $w \prec p$ holds $\mathrm{succ}\, v = \mathrm{succ}\, w$.

(18) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_1 such that $v = w$ and p and w are not comparable holds $\mathrm{succ}\, v = \mathrm{succ}\, w$.

(19) For all trees Z_1, Z_2 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of $Z_1(p/Z_2)$ and for every element w of Z_2 such that $v = p \frown w$ holds $\mathrm{succ}\, v \approx \mathrm{succ}\, w$.

(20) For every tree Z_1 and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in Z_1$ for every element v of Z_1 and for every element w of $Z_1 \upharpoonright p$ such that $v = p \frown w$ holds $\mathrm{succ}\, v \approx \mathrm{succ}\, w$.

(21) For every tree Z and for every element p of Z such that Z is finite holds $\mathrm{succ}\, p$ is finite.

(22) For every tree Z such that Z is finite and the branch degree of the root of $Z = 0$ holds $\mathrm{card}\, Z = 1$ and $Z = \{\varepsilon\}$.

(23) For every tree Z such that Z is finite and the branch degree of the root of $Z = 1$ holds succ(the root of Z) $= \{\langle 0 \rangle\}$.

(24) For every tree Z such that Z is finite and the branch degree of the root of $Z = 2$ holds succ(the root of Z) $= \{\langle 0 \rangle, \langle 1 \rangle\}$.

In the sequel s', w' will be elements of $\mathbb{N}^*$. One can prove the following propositions:

(25) For every tree Z and for every element o of Z such that $o \neq$ the root of Z holds $Z \upharpoonright o \approx \{o \frown s' : o \frown s' \in Z\}$ and the root of $Z \notin \{o \frown w' : o \frown w' \in Z\}$.

(26) For every tree Z and for every element o of Z such that $o \neq$ the root of Z and Z is finite holds $\mathrm{card}(Z \upharpoonright o) < \mathrm{card}\, Z$.

(27) For every tree Z and for every element z of Z such that succ(the root of Z) $= \{z\}$ and Z is finite holds $Z =$ (the elementary tree of 1)$(\langle 0 \rangle/(Z \upharpoonright z))$.

(28) For every tree Z decorated by D and for every element z of $\mathrm{dom}\, Z$ such that succ(the root of $\mathrm{dom}\, Z$) $= \{z\}$ and $\mathrm{dom}\, Z$ is finite holds $Z =$ (the elementary tree of $1 \longmapsto$ the root of Z)$(\langle 0 \rangle/(Z \upharpoonright z))$.

(29) For every tree Z and for all elements x_1, x_2 of Z such that Z is finite and $x_1 = \langle 0 \rangle$ and $x_2 = \langle 1 \rangle$ and succ(the root of Z) $= \{x_1, x_2\}$ holds $Z =$ (the elementary tree of 2)$(\langle 0 \rangle/(Z \upharpoonright x_1))(\langle 1 \rangle/(Z \upharpoonright x_2))$.

(30) Let Z be a tree decorated by D. Then for all elements x_1, x_2 of $\operatorname{dom} Z$ such that $\operatorname{dom} Z$ is finite and $x_1 = \langle 0 \rangle$ and $x_2 = \langle 1 \rangle$ and succ(the root of $\operatorname{dom} Z) = \{x_1, x_2\}$ holds $Z =$ (the elementary tree of $2 \longmapsto$ the root of $Z)(\langle 0 \rangle / (Z \restriction x_1))(\langle 1 \rangle / (Z \restriction x_2))$.

The non-empty set $\mathcal{V}$ is defined by:

(Def.3) $\mathcal{V} = [\!:\{3\}, \mathbb{N}:\!]$.

A variable is an element of $\mathcal{V}$.

The non-empty set $\mathcal{C}$ is defined as follows:

(Def.4) $\mathcal{C} = [\!:\{0,1,2\}, \mathbb{N}:\!]$.

A conective is an element of $\mathcal{C}$.

One can prove the following proposition

(31) $\mathcal{C} \cap \mathcal{V} = \emptyset$.

In the sequel p, q denote variables. Let T be a tree, and let v be an element of T. Then the branch degree of v is a natural number.

Let D be a non-empty set. A non-empty set is called a non-empty set of trees decorated by D if:

(Def.5) for every x such that $x \in$ it holds x is a tree decorated by D.

Let D_0 be a non-empty set, and let D be a non-empty set of trees decorated by D_0. We see that the element of D is a tree decorated by D_0.

The non-empty set WFF of trees decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set $:\!]$ is defined by the condition (Def.6).

(Def.6) Let x be a tree decorated by $[\!:\mathbb{N}, \mathbb{N}$ **qua** a non-empty set $:\!]$. Then $x \in$ WFF if and only if the following conditions are satisfied:

(i) $\operatorname{dom} x$ is finite,

(ii) for every element v of $\operatorname{dom} x$ holds the branch degree of $v \leq 2$ but if the branch degree of $v = 0$, then $x(v) = \langle 0, 0 \rangle$ or there exists k such that $x(v) = \langle 3, k \rangle$ but if the branch degree of $v = 1$, then $x(v) = \langle 1, 0 \rangle$ or $x(v) = \langle 1, 1 \rangle$ but if the branch degree of $v = 2$, then $x(v) = \langle 2, 0 \rangle$.

A MP-formula is an element of WFF.

In the sequel A, A_1, B, B_1, C denote MP-formulae. Let us consider A, and let a be an element of $\operatorname{dom} A$. Then $A \restriction a$ is a MP-formula.

Let a be an element of $\mathcal{C}$. The functor Arity(a) yielding a natural number is defined by:

(Def.7) $\text{Arity}(a) = a_{\mathbf{1}}$.

Let D be a non-empty set, and let T, T_1 be trees decorated by D, and let p be a finite sequence of elements of $\mathbb{N}$. Let us assume that $p \in \operatorname{dom} T$. The functor $T(p \leftarrow T_1)$ yields a tree decorated by D and is defined by:

(Def.8) $T(p \leftarrow T_1) = T(p/T_1)$.

The following propositions are true:

(32) (The elementary tree of $1 \longmapsto \langle 1, 0 \rangle)(\langle 0 \rangle / A)$ is a MP-formula.

(33) (The elementary tree of $1 \longmapsto \langle 1, 1\rangle)(\langle 0\rangle/A)$ is a MP-formula.

(34) (The elementary tree of $2 \longmapsto \langle 2, 0\rangle)(\langle 0\rangle/A)(\langle 1\rangle/B)$ is a MP-formula.

We now define three new functors. Let us consider A. The functor $\neg A$ yields a MP-formula and is defined as follows:

(Def.9) $\neg A =$ (the elementary tree of $1 \longmapsto \langle 1, 0\rangle)(\langle 0\rangle/A)$.

The functor $\Box A$ yields a MP-formula and is defined as follows:

(Def.10) $\Box A =$ (the elementary tree of $1 \longmapsto \langle 1, 1\rangle)(\langle 0\rangle/A)$.

Let us consider B. The functor $A \wedge B$ yielding a MP-formula is defined as follows:

(Def.11) $A \wedge B =$ (the elementary tree of $2 \longmapsto \langle 2, 0\rangle)(\langle 0\rangle/A)(\langle 1\rangle/B)$.

We now define three new functors. Let us consider A. The functor $\Diamond A$ yields a MP-formula and is defined as follows:

(Def.12) $\Diamond A = \neg\Box\neg A$.

Let us consider B. The functor $A \vee B$ yields a MP-formula and is defined as follows:

(Def.13) $A \vee B = \neg(\neg A \wedge \neg B)$.

The functor $A \Rightarrow B$ yields a MP-formula and is defined by:

(Def.14) $A \Rightarrow B = \neg(A \wedge \neg B)$.

The following propositions are true:

(35) The elementary tree of $0 \longmapsto \langle 3, n\rangle$ is a MP-formula.

(36) The elementary tree of $0 \longmapsto \langle 0, 0\rangle$ is a MP-formula.

Let us consider p. The functor $^{@}p$ yields a MP-formula and is defined by:

(Def.15) $^{@}p =$ the elementary tree of $0 \longmapsto p$.

We now state four propositions:

(37) If $^{@}p = {}^{@}q$, then $p = q$.

(38) If $\neg A = \neg B$, then $A = B$.

(39) If $\Box A = \Box B$, then $A = B$.

(40) If $A \wedge B = A_1 \wedge B_1$, then $A = A_1$ and $B = B_1$.

The MP-formula VERUM is defined by:

(Def.16) VERUM = the elementary tree of $0 \longmapsto \langle 0, 0\rangle$.

Next we state several propositions:

(41) $\operatorname{card}\operatorname{dom} A \neq 0$.

(42) If $\operatorname{card}\operatorname{dom} A = 1$, then $A =$ VERUM or there exists p such that $A = {}^{@}p$.

(43) If $\operatorname{card}\operatorname{dom} A \geq 2$, then there exists B such that $A = \neg B$ or $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(44) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom} \neg A$.

(45) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom} \Box A$.

(46) $\operatorname{card}\operatorname{dom} A < \operatorname{card}\operatorname{dom}(A \wedge B)$ and $\operatorname{card}\operatorname{dom} B < \operatorname{card}\operatorname{dom}(A \wedge B)$.

We now define four new attributes. A MP-formula is atomic if:

(Def.17) there exists p such that it $= {}^{@}p$.

A MP-formula is negative if:

(Def.18) there exists A such that it $= \neg A$.

A MP-formula is necessitive if:

(Def.19) there exists A such that it $= \Box A$.

A MP-formula is conjunctive if:

(Def.20) there exist A, B such that it $= A \wedge B$.

The scheme *MP_Ind* deals with a unary predicate $\mathcal{P}$, and states that:

for every element A of WFF holds $\mathcal{P}[A]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\text{VERUM}]$,
- for every variable p holds $\mathcal{P}[{}^{@}p]$,
- for every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\neg A]$,
- for every element A of WFF such that $\mathcal{P}[A]$ holds $\mathcal{P}[\Box A]$,
- for all elements A, B of WFF such that $\mathcal{P}[A]$ and $\mathcal{P}[B]$ holds $\mathcal{P}[A \wedge B]$.

The following propositions are true:

(47) For every element A of WFF holds $A =$ VERUM or A is a MP-formula or A is a MP-formula or A is a MP-formula or A is a MP-formula.

(48) $A =$ VERUM or there exists p such that $A = {}^{@}p$ or there exists B such that $A = \neg B$ or there exists B such that $A = \Box B$ or there exist B, C such that $A = B \wedge C$.

(49) ${}^{@}p \neq \neg A$ and ${}^{@}p \neq \Box A$ and ${}^{@}p \neq A \wedge B$.

(50) $\neg A \neq \Box B$ and $\neg A \neq B \wedge C$.

(51) $\Box A \neq B \wedge C$.

(52) VERUM $\neq {}^{@}p$ and VERUM $\neq \neg A$ and VERUM $\neq \Box A$ and VERUM $\neq A \wedge B$.

The scheme *MP_Func_Ex* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$ and states that:

there exists a function f from WFF into $\mathcal{A}$ such that $f(\text{VERUM}) = \mathcal{B}$ and for every variable p holds $f({}^{@}p) = \mathcal{F}(p)$ and for every element A of WFF and for every element d of $\mathcal{A}$ such that $f(A) = d$ holds $f(\neg A) = \mathcal{G}(d)$ and for every element A of WFF and for every element d of $\mathcal{A}$ such that $f(A) = d$ holds $f(\Box A) = \mathcal{H}(d)$ and for all elements A, B of WFF and for all elements d_1, d_2 of $\mathcal{A}$ such that $d_1 = f(A)$ and $d_2 = f(B)$ holds $f(A \wedge B) = \mathcal{I}(d_1, d_2)$

for all values of the parameters.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[4] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[17] Wojciech Zielonka. Preliminaries to the Lambek calculus. *Formalized Mathematics*, 2(**3**):413–418, 1991.

Received September 30, 1990