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Natural Transformations. Discrete Categories

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Summary. We present well known concepts of category theory: natural transofmations and functor categories, and prove propositions related to. Because of the formalization it proved to be convenient to introduce some auxiliary notions, for instance: transformations. We mean by a transformation of a functor F to a functor G, both covariant functors from A to B, a function mapping the objects of A to the morphisms of B and assigning to an object a of A an element of $\mathrm{Hom}(F(a), G(a))$. The material included roughly corresponds to that presented on pages 18,129–130,137–138 of the monography ([10]). We also introduce discrete categories and prove some propositions to illustrate the concepts introduced.

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The articles [12], [13], [9], [3], [7], [4], [2], [6], [1], [11], [5], and [8] provide the terminology and notation for this paper.

Preliminaries

For simplicity we follow a convention: A_1, A_2, B_1, B_2 are non-empty sets, f is a function from A_1 into B_1, g is a function from A_2 into B_2, Y_1 is a non-empty subset of A_1, and Y_2 is a non-empty subset of A_2. Let A_1, A_2 be non-empty sets, and let Y_1 be a non-empty subset of A_1, and let Y_2 be a non-empty subset of A_2. Then $[\!: Y_1, Y_2 :\!]$ is a non-empty subset of $[\!: A_1, A_2 :\!]$.

Let us consider A_1, B_1, f, Y_1. Then $f \restriction Y_1$ is a function from Y_1 into B_1.

We now state the proposition

(1) $[\!: f, g :\!] \restriction [\!: Y_1, Y_2 :\!] = [\!: f \restriction Y_1, g \restriction Y_2 :\!]$.

Let A, B be non-empty sets, and let A_1 be a non-empty subset of A, and let B_1 be a non-empty subset of B, and let f be a partial function from $[\!: A_1, A_1 :\!]$

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to A_1, and let g be a partial function from $[: B_1, B_1 :]$ to B_1. Then $|:f, g:|$ is a partial function from $[: [: A_1, B_1 :], [: A_1, B_1 :] :]$ to $[: A_1, B_1 :]$.

One can prove the following proposition

(2) Let f be a partial function from $[: A_1, A_1 :]$ to A_1. Let g be a partial function from $[: A_2, A_2 :]$ to A_2. Then for every partial function F from $[: Y_1, Y_1 :]$ to Y_1 such that $F = f \restriction [: Y_1, Y_1 :]$ for every partial function G from $[: Y_2, Y_2 :]$ to Y_2 such that $G = g \restriction [: Y_2, Y_2 :]$ holds $|:F, G:| = |:f, g:| \restriction [: [: Y_1, Y_2 :], [: Y_1, Y_2 :] :]$.

We adopt the following convention: A, B, C will be categories, F, F_1, F_2, F_3 will be functors from A to B, and G will be a functor from B to C. In this article we present several logical schemes. The scheme *M_Choice* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set and states that:

there exists a function t from $\mathcal{A}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds $t(a) \in \mathcal{F}(a)$

provided the following requirement is met:

- for every element a of $\mathcal{A}$ holds $\mathcal{B}$ meets $\mathcal{F}(a)$.

The scheme *LambdaT* concerns a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the following requirement is met:

- for every element x of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

We now state the proposition

(3) For every object a of A and for every morphism m from a to a holds $m \in \hom(a, a)$.

In the sequel m, o will be arbitrary. One can prove the following propositions:

(4) For all morphisms f, g of $\dot{\circlearrowleft}(o, m)$ holds $f = g$.

(5) For every object a of A holds $\langle\langle \mathrm{id}_a, \mathrm{id}_a \rangle, \mathrm{id}_a \rangle \in$ the composition of A.

(6) The composition of $\dot{\circlearrowleft}(o, m) = \{\langle\langle m, m\rangle, m\rangle\}$.

(7) For every object a of A holds $\dot{\circlearrowleft}(a, \mathrm{id}_a)$ is a subcategory of A.

(8) For every subcategory C of A holds the dom-map of $C =$ (the dom-map of A)$\restriction$ the morphisms of C and the cod-map of $C =$ (the cod-map of A)$\restriction$ the morphisms of C and the composition of $C =$ (the composition of A)$\restriction [:$ the morphisms of C, the morphisms of $C :]$ and the id-map of $C =$ (the id-map of A)$\restriction$ the objects of C.

(9) Let O be a non-empty subset of the objects of A. Let M be a non-empty subset of the morphisms of A. Let D_1, C_1 be functions from M into O. Suppose $D_1 =$ (the dom-map of A)$\restriction M$ and $C_1 =$ (the cod-map of A)$\restriction M$. Then for every partial function C_2 from $[: M, M$ **qua** a non-empty set $:]$ to M such that $C_2 =$ (the composition of A) $\restriction [: M, M :]$ for every function I_1 from O into M such that $I_1 =$ (the id-map of A) $\restriction O$ holds $\langle O, M, D_1, C_1, C_2, I_1\rangle$ is a subcategory of A.

(10) For every subcategory A of C such that the objects of $A =$ the objects of C and the morphisms of $A =$ the morphisms of C holds $A = C$.

Application of a functor to a morphism

Let us consider A, B, F, and let a, b be objects of A satisfying the condition: $\hom(a,b) \neq \emptyset$. Let f be a morphism from a to b. The functor $F(f)$ yields a morphism from $F(a)$ to $F(b)$ and is defined by:

(Def.1) $F(f) = F(f)$.

One can prove the following propositions:

(11) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ for every morphism f from a to b holds $(G \cdot F)(f) = G(F(f))$.

(12) For all functors F_1, F_2 from A to B such that for all objects a, b of A such that $\hom(a,b) \neq \emptyset$ for every morphism f from a to b holds $F_1(f) = F_2(f)$ holds $F_1 = F_2$.

(13) For all objects a, b, c of A such that $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ for every morphism f from a to b and for every morphism g from b to c holds $F(g \cdot f) = F(g) \cdot F(f)$.

(14) For every object c of A and for every object d of B such that $F(\mathrm{id}_c) = \mathrm{id}_d$ holds $F(c) = d$.

(15) For every object a of A holds $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

(16) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ for every morphism f from a to b holds $\mathrm{id}_A(f) = f$.

(17) For all objects a, b, c, d of A such that $\hom(a,b)$ meets $\hom(c,d)$ holds $a = c$ and $b = d$.

Transformations

Let us consider A, B, F_1, F_2. We say that F_1 is transformable to F_2 if and only if:

(Def.2) for every object a of A holds $\hom(F_1(a), F_2(a)) \neq \emptyset$.

One can prove the following propositions:

(18) F is transformable to F.

(19) If F is transformable to F_1 and F_1 is transformable to F_2, then F is transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. A function from the objects of A into the morphisms of B is said to be a transformation from F_1 to F_2 if:

(Def.3) for every object a of A holds it(a) is a morphism from $F_1(a)$ to $F_2(a)$.

Let us consider A, B, and let F be a functor from A to B. The functor id_F yields a transformation from F to F and is defined as follows:

(Def.4) for every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2, and let a be an object of A. The functor $t(a)$ yields a morphism from $F_1(a)$ to $F_2(a)$ and is defined by:

(Def.5) $t(a) = t(a)$.

Let us consider A, B, F, F_1, F_2. Let us assume that F is transformable to F_1 and F_1 is transformable to F_2. Let t_1 be a transformation from F to F_1, and let t_2 be a transformation from F_1 to F_2. The functor $t_2{\circ}t_1$ yields a transformation from F to F_2 and is defined by:

(Def.6) for every object a of A holds $(t_2{\circ}t_1)(a) = t_2(a) \cdot t_1(a)$.

The following propositions are true:

(20) If F_1 is transformable to F_2, then for all transformations t_1, t_2 from F_1 to F_2 such that for every object a of A holds $t_1(a) = t_2(a)$ holds $t_1 = t_2$.

(21) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

(22) If F_1 is transformable to F_2, then for every transformation t from F_1 to F_2 holds $\mathrm{id}_{F_2}{\circ}t = t$ and $t{\circ}\,\mathrm{id}_{F_1} = t$.

(23) If F is transformable to F_1 and F_1 is transformable to F_2 and F_2 is transformable to F_3, then for every transformation t_1 from F to F_1 and for every transformation t_2 from F_1 to F_2 and for every transformation t_3 from F_2 to F_3 holds $t_3{\circ}t_2{\circ}t_1 = t_3{\circ}(t_2{\circ}t_1)$.

Natural transformations

Let us consider A, B, F_1, F_2. We say that F_1 is naturally transformable to F_2 if and only if:

(Def.7) F_1 is transformable to F_2 and there exists a transformation t from F_1 to F_2 such that for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b holds $t(b) \cdot F_1(f) = F_2(f) \cdot t(a)$.

Next we state two propositions:

(24) F is naturally transformable to F.

(25) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2, then F is naturally transformable to F_2.

Let us consider A, B, F_1, F_2. Let us assume that F_1 is naturally transformable to F_2. A transformation from F_1 to F_2 is called a natural transformation from F_1 to F_2 if:

(Def.8) for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b holds $\mathrm{it}(b) \cdot F_1(f) = F_2(f) \cdot \mathrm{it}(a)$.

Let us consider A, B, F. Then id_F is a natural transformation from F to F.

Let us consider A, B, F, F_1, F_2. satisfying the conditions: F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1, and let t_2 be a natural transformation from F_1 to F_2. The functor $t_2{\circ}t_1$ yields a natural transformation from F to F_2 and is defined by:

(Def.9) $t_2{\circ}t_1 = t_2{\circ}t_1$.

One can prove the following proposition

(26) If F_1 is naturally transformable to F_2, then for every natural transformation t from F_1 to F_2 holds $\mathrm{id}_{F_2}{\circ}t = t$ and $t{\circ}\,\mathrm{id}_{F_1} = t$.

In the sequel t denotes a natural transformation from F to F_1 and t_1 denotes a natural transformation from F_1 to F_2. Next we state two propositions:

(27) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2, then for every natural transformation t_1 from F to F_1 and for every natural transformation t_2 from F_1 to F_2 and for every object a of A holds $(t_2{\circ}t_1)(a) = t_2(a) \cdot t_1(a)$.

(28) If F is naturally transformable to F_1 and F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3, then for every natural transformation t_3 from F_2 to F_3 holds $t_3{\circ}t_1{\circ}t = t_3{\circ}(t_1{\circ}t)$.

Let us consider A, B, F_1, F_2. A transformation from F_1 to F_2 is invertible if:

(Def.10) for every object a of A holds $\mathrm{it}(a)$ is invertible.

We now define two new predicates. Let us consider A, B, F_1, F_2. We say that F_1, F_2 are naturally equivalent if and only if:

(Def.11) F_1 is naturally transformable to F_2 and there exists a natural transformation t from F_1 to F_2 such that t is invertible.

We write $F_1 \cong F_2$ if and only if F_1, F_2 are naturally equivalent.

One can prove the following proposition

(29) $F \cong F$.

Let us consider A, B, F_1, F_2. satisfying the condition: F_1 is transformable to F_2. Let t_1 be a transformation from F_1 to F_2 satisfying the condition: t_1 is invertible. The functor ${t_1}^{-1}$ yielding a transformation from F_2 to F_1 is defined as follows:

(Def.12) for every object a of A holds ${t_1}^{-1}(a) = t_1(a)^{-1}$.

Let us consider A, B, F_1, F_2, t_1. satisfying the conditions: F_1 is naturally transformable to F_2 and t_1 is invertible. The functor ${t_1}^{-1}$ yielding a natural transformation from F_2 to F_1 is defined by:

(Def.13) ${t_1}^{-1} = (t_1$ **qua** a transformation from F_1 to $F_2)^{-1}$.

Next we state three propositions:

(30) For all A, B, F_1, F_2, t_1 such that F_1 is naturally transformable to F_2 and t_1 is invertible for every object a of A holds ${t_1}^{-1}(a) = t_1(a)^{-1}$.

(31) If $F_1 \cong F_2$, then $F_2 \cong F_1$.

(32) If $F_1 \cong F_2$ and $F_2 \cong F_3$, then $F_1 \cong F_3$.

Let us consider A, B, F_1, F_2. Let us assume that F_1, F_2 are naturally equivalent. A natural transformation from F_1 to F_2 is called a natural equivalence of F_1 and F_2 if:

(Def.14) it is invertible.

We now state two propositions:

(33) id_F is a natural equivalence of F and F.

(34) If $F_1 \cong F_2$ and $F_2 \cong F_3$, then for every natural equivalence t of F_1 and F_2 and for every natural equivalence t' of F_2 and F_3 holds $t' \circ t$ is a natural equivalence of F_1 and F_3.

Functor category

Let us consider A, B. A non-empty set is called a set of natural transformations from A to B if:

(Def.15) for an arbitrary x such that $x \in$ it there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let us consider A, B. The functor $\mathrm{NatTrans}(A, B)$ yielding a set of natural transformations from A to B is defined as follows:

(Def.16) for an arbitrary x holds $x \in \mathrm{NatTrans}(A, B)$ if and only if there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that $x = \langle\langle F_1, F_2\rangle, t\rangle$ and F_1 is naturally transformable to F_2.

Let A_1, B_1, A_2, B_2 be non-empty sets, and let f_1 be a function from A_1 into B_1, and let f_2 be a function from A_2 into B_2. Let us note that one can characterize the predicate $f_1 = f_2$ by the following (equivalent) condition:

(Def.17) $A_1 = A_2$ and for every element a of A_1 holds $f_1(a) = f_2(a)$.

The following two propositions are true:

(35) F_1 is naturally transformable to F_2 if and only if $\langle\langle F_1, F_2\rangle, t_1\rangle \in \mathrm{NatTrans}(A, B)$.

(36) $\langle\langle F, F\rangle, \mathrm{id}_F\rangle \in \mathrm{NatTrans}(A, B)$.

Let us consider A, B. The functor B^A yielding a category is defined by the conditions (Def.18).

(Def.18) (i) The objects of $B^A = \mathrm{Funct}(A, B)$,

(ii) the morphisms of $B^A = \mathrm{NatTrans}(A, B)$,

(iii) for every morphism f of B^A holds $\mathrm{dom}\, f = (f_{\mathbf{1}})_{\mathbf{1}}$ and $\mathrm{cod}\, f = (f_{\mathbf{1}})_{\mathbf{2}}$,

(iv) for all morphisms f, g of B^A such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\langle g, f\rangle \in$ dom (the composition of B^A),

(v) for all morphisms f, g of B^A such that $\langle g, f\rangle \in$ dom (the composition of B^A) there exist F, F_1, F_2, t, t_1 such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ and (the composition of B^A)$(\langle g, f\rangle) = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$,

(vi) for every object a of B^A and for every F such that $F = a$ holds $\mathrm{id}_a = \langle\langle F, F\rangle, \mathrm{id}_F\rangle$.

We now state several propositions:

(37) The objects of $B^A = \mathrm{Funct}(A, B)$.

(38) The morphisms of $B^A = \mathrm{NatTrans}(A, B)$.

(39) For every morphism f of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ holds $\mathrm{dom}\, f = F$ and $\mathrm{cod}\, f = F_1$.

(40) For all objects a, b of B^A and for every morphism f from a to b such that $\hom(a, b) \neq \emptyset$ there exist F, F_1, t such that $a = F$ and $b = F_1$ and $f = \langle\langle F, F_1\rangle, t\rangle$.

(41) For every natural transformation t' from F_2 to F_3 and for all morphisms f, g of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_2, F_3\rangle, t'\rangle$ holds $\langle g, f\rangle \in$ dom (the composition of B^A) if and only if $F_1 = F_2$.

(42) For all morphisms f, g of B^A such that $f = \langle\langle F, F_1\rangle, t\rangle$ and $g = \langle\langle F_1, F_2\rangle, t_1\rangle$ holds $g \cdot f = \langle\langle F, F_2\rangle, t_1 \circ t\rangle$.

(43) For every object a of B^A and for every F such that $F = a$ holds $\mathrm{id}_a = \langle\langle F, F\rangle, \mathrm{id}_F\rangle$.

Discrete categories

A category is discrete if:

(Def.19) for every morphism f of it there exists an object a of it such that $f = \mathrm{id}_a$.

One can prove the following propositions:

(44) For every discrete category A and for every object a of A holds $\hom(a, a) = \{\mathrm{id}_a\}$.

(45) A is discrete if and only if for every object a of A holds $\hom(a, a)$ is finite and $\mathrm{card}\,\hom(a, a) = 1$ and for every object b of A such that $a \neq b$ holds $\hom(a, b) = \emptyset$.

(46) $\dot{\circlearrowright}(o, m)$ is discrete.

(47) For every discrete category A and for every subcategory C of A holds C is discrete.

(48) If A is discrete and B is discrete, then $[\!: A, B :\!]$ is discrete.

(49) For every discrete category A and for every category B and for all functors F_1, F_2 from B to A such that F_1 is transformable to F_2 holds $F_1 = F_2$.

(50) For every discrete category A and for every category B and for every functor F from B to A and for every transformation t from F to F holds $t = \mathrm{id}_F$.

(51) If A is discrete, then A^B is discrete.

Let us consider C. The functor $\mathrm{IdCat}\, C$ yields a discrete subcategory of C and is defined as follows:

(Def.20) the objects of $\mathrm{IdCat}\, C$ = the objects of C and the morphisms of $\mathrm{IdCat}\, C = \{\mathrm{id}_a\}$,
where a ranges over objects of C.

Next we state four propositions:

(52) If C is discrete, then $\mathrm{IdCat}\, C = C$.

(53) $\mathrm{IdCat}\,\mathrm{IdCat}\,C = \mathrm{IdCat}\,C$.

(54) $\mathrm{IdCat}\,\dot{\circlearrowright}(o, m) = \dot{\circlearrowright}(o, m)$.

(55) $\mathrm{IdCat}[\!:A,\ B:\!] = [\!:\mathrm{IdCat}\,A,\ \mathrm{IdCat}\,B:\!]$.

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