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# Paracompact and Metrizable Spaces

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**Summary.** We give an example of a compact space. Next we define a locally finite subset family of topological spaces and paracompact topological spaces. An open sets family of a metric space is defined next and it has been shown that the metric space with any open sets family is a topological space. Next we define metrizable space.

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The papers [15], [5], [6], [11], [10], [12], [13], [18], [8], [17], [9], [7], [16], [3], [2], [1], [4], and [14] provide the terminology and notation for this paper. In the sequel $P_1$ denotes a metric space, $x$ denotes an element of the carrier of $P_1$, and $r$, $p$ denote real numbers. Next we state the proposition

(1) If $r \leq p$ and $r > 0$, then $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, p)$.

For simplicity we adopt the following convention: $T$ will be a topological space, $x$ will be a point of $T$, $W$, $A$ will be subsets of $T$, and $F_1$ will be a family of subsets of $T$. One can prove the following four propositions:

(2) $\overline{A} \neq \emptyset$ if and only if $A \neq \emptyset$.

(3) If $\overline{A} = \emptyset$, then $A = \emptyset$.

(4) $\overline{A}$ is closed.

(5) If $F_1$ is a cover of $T$, then for every $x$ there exists $W$ such that $x \in W$ and $W \in F_1$.

Let $X$ be arbitrary. Then $\{X\}$ is a non-empty set. Then $2^X$ is a non-empty family of subsets of $X$.

Let $a$ be arbitrary. The functor $\{a\}_{\mathrm{top}}$ yields a topological space and is defined by:

(Def.1) $\{a\}_{\mathrm{top}} = \langle \{a\}, 2^{\{a\}} \rangle$.

In the sequel $a$ is arbitrary. We now state four propositions:

(6) $\{a\}_{\mathrm{top}} = \langle \{a\}, 2^{\{a\}} \rangle$.




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(7) The topology of $\{a\}_{\rm top} = 2^{\{a\}}$.

(8) The carrier of $\{a\}_{\rm top} = \{a\}$.

(9) $\{a\}_{\rm top}$ is compact.

Let us consider $T$, $x$. Then $\{x\}$ is a subset of $T$.

We now state the proposition

(10) If $T$ is a $T_2$ space, then $\{x\}$ is closed.

For simplicity we follow the rules: $T$ will be a topological space, $x$ will be a point of $T$, $Z$, $V$, $W$, $Y$, $A$, $B$ will be subsets of $T$, and $F_1$, $G_1$ will be families of subsets of $T$. Let us consider $T$. A family of subsets of $T$ is locally finite if:

(Def.2) for every $x$ there exists $W$ such that $x \in W$ and $W$ is open and $\{V : V \in \text{it} \wedge V \cap W \neq \emptyset\}$ is finite.

Next we state three propositions:

(11) For every $W$ holds $\{V : V \in F_1 \wedge V \cap W \neq \emptyset\} \subseteq F_1$.

(12) If $F_1 \subseteq G_1$ and $G_1$ is locally finite, then $F_1$ is locally finite.

(13) If $F_1$ is finite, then $F_1$ is locally finite.

Let us consider $T$, $F_1$. The functor $\operatorname{clf} F_1$ yielding a family of subsets of $T$ is defined by:

(Def.3) $Z \in \operatorname{clf} F_1$ if and only if there exists $W$ such that $Z = \overline{W}$ and $W \in F_1$.

Next we state several propositions:

(14) $\operatorname{clf} F_1$ is closed.

(15) If $F_1 = \emptyset$, then $\operatorname{clf} F_1 = \emptyset$.

(16) If $F_1 = \{V\}$, then $\operatorname{clf} F_1 = \{\overline{V}\}$.

(17) If $F_1 \subseteq G_1$, then $\operatorname{clf} F_1 \subseteq \operatorname{clf} G_1$.

(18) $\operatorname{clf}(F_1 \cup G_1) = \operatorname{clf} F_1 \cup \operatorname{clf} G_1$.

Next we state two propositions:

(19) If $F_1$ is finite, then $\overline{\bigcup F_1} = \bigcup \operatorname{clf} F_1$.

(20) $F_1$ is finer than $\operatorname{clf} F_1$.

The scheme *Lambda1top* deals with a topological space $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a family $\mathcal{C}$ of subsets of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$ and states that:

there exists a function $f$ from $\mathcal{B}$ into $\mathcal{C}$ such that for every subset $Z$ of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $f(Z) = \mathcal{F}(Z)$

provided the following condition is satisfied:

- for every subset $Z$ of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $\mathcal{F}(Z) \in \mathcal{C}$.

Next we state four propositions:

(21) If $F_1$ is locally finite, then $\operatorname{clf} F_1$ is locally finite.

(22) $\bigcup F_1 \subseteq \bigcup \operatorname{clf} F_1$.

(23) If $F_1$ is locally finite, then $\overline{\bigcup F_1} = \bigcup \operatorname{clf} F_1$.

(24) If $F_1$ is locally finite and $F_1$ is closed, then $\bigcup F_1$ is closed.

A topological space is paracompact if:

(Def.4) for every family $F_1$ of subsets of it such that $F_1$ is a cover of it and $F_1$ is open there exists a family $G_1$ of subsets of it such that $G_1$ is open and $G_1$ is a cover of it and $G_1$ is finer than $F_1$ and $G_1$ is locally finite.

The following propositions are true:

(25) If $T$ is compact, then $T$ is paracompact.

(26) Suppose $T$ is paracompact and $A$ is closed and $B$ is closed and $A$ misses $B$ and for every $x$ such that $x \in B$ there exist $V$, $W$ such that $V$ is open and $W$ is open and $A \subseteq V$ and $x \in W$ and $V$ misses $W$. Then there exist $Y$, $Z$ such that $Y$ is open and $Z$ is open and $A \subseteq Y$ and $B \subseteq Z$ and $Y$ misses $Z$.

(27) If $T$ is a $\mathrm{T}_2$ space and $T$ is paracompact, then $T$ is a $\mathrm{T}_3$ space.

(28) If $T$ is a $\mathrm{T}_2$ space and $T$ is paracompact, then $T$ is a $\mathrm{T}_4$ space.

For simplicity we follow a convention: $P_1$ will denote a metric space, $x$, $y$, $z$ will denote elements of the carrier of $P_1$, $r$, $p$, $q$ will denote real numbers, and $V$, $W$ will denote subsets of the carrier of $P_1$. Let us consider $P_1$. The open set family of $P_1$ yielding a family of subsets of the carrier of $P_1$ is defined as follows:

(Def.5) for every $V$ holds $V \in$the open set family of $P_1$ if and only if for every $x$ such that $x \in V$ there exists $r$ such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

One can prove the following propositions:

(29) For every $x$ there exists $r$ such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq$ the carrier of $P_1$.

(30) If $y \in \mathrm{Ball}(x, r)$, then there exists $p$ such that $p > 0$ and $\mathrm{Ball}(y, p) \subseteq \mathrm{Ball}(x, r)$.

(31) If $y \in \mathrm{Ball}(x, r) \cap \mathrm{Ball}(z, p)$, then there exists $q$ such that $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(x, r)$ and $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(z, p)$.

(32) For every $V$ holds $V \in$ the open set family of $P_1$ if and only if for every $x$ such that $x \in V$ there exists $r$ such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

(33) For all $x$, $r$ holds $\mathrm{Ball}(x, r) \in$ the open set family of $P_1$.

(34) The carrier of $P_1 \in$ the open set family of $P_1$.

(35) For all $V$, $W$ such that $V \in$ the open set family of $P_1$ and $W \in$ the open set family of $P_1$ holds $V \cap W \in$ the open set family of $P_1$.

(36) For every family $A$ of subsets of the carrier of $P_1$ such that $A \subseteq$ the open set family of $P_1$ holds $\bigcup A \in$ the open set family of $P_1$.

(37) $\langle$The carrier of $P_1$,the open set family of $P_1\rangle$ is a topological space.

Let us consider $P_1$. The functor $P_{1\mathrm{top}}$ yielding a topological space is defined as follows:

(Def.6) $P_{1\mathrm{top}} = \langle$ the carrier of $P_1$,the open set family of $P_1\rangle$.

We now state the proposition

(38) $P_{1\mathrm{top}}$ is a $\mathrm{T}_2$ space.

Let $D$ be a non-empty set, and let $f$ be a function from $[\!: D, D :\!]$ into $\mathbb{R}$. We say that $f$ is a metric of $D$ if and only if:

(Def.7) for all elements $a$, $b$, $c$ of $D$ holds $f(a, b) = 0$ if and only if $a = b$ but $f(a, b) = f(b, a)$ and $f(a, c) \leq f(a, b) + f(b, c)$.

We now state two propositions:

(39) For every non-empty set $D$ and for every function $f$ from $[: D, D :]$ into $\mathbb{R}$ holds $f$ is a metric of $D$ if and only if $\langle D, f \rangle$ is a metric space.

(40) For every metric space $M_1$ holds the distance of $M_1$ is a metric of the carrier of $M_1$.

Let $D$ be a non-empty set, and let $f$ be a function from $[: D, D :]$ into $\mathbb{R}$. Let us assume that $f$ is a metric of $D$. The functor $\mathrm{MetrSp}(D, f)$ yielding a metric space is defined by:

(Def.8) $\mathrm{MetrSp}(D, f) = \langle D, f \rangle$.

A topological space is metrizable if:

(Def.9) there exists a function $f$ from $[:$ the carrier of it, the carrier of it $:]$ into $\mathbb{R}$ such that $f$ is a metric of the carrier of it and the open set family of $\mathrm{MetrSp}((\text{the carrier of it}), f) =$ the topology of it.

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