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Quadratic Inequalities

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Summary. Consider a quadratic trinomial of the form $P(x) = ax^2 + bx + c$, where $a \neq 0$. The determinat of the equation $P(x) = 0$ is of the form $\Delta(a, b, c) = b^2 - 4ac$. We prove several quadratic inequalities when $\Delta(a, b, c) < 0$, $\Delta(a, b, c) = 0$ and $\Delta(a, b, c) > 0$.

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The articles [3], [1], [2], and [4] provide the terminology and notation for this paper. In the sequel x is a real number and a, b, c are real numbers. Let us consider a, b, c. The functor $\Delta(a, b, c)$ yielding a real number is defined as follows:

(Def.1) $\Delta(a, b, c) = b^{\mathbf{2}} - 4 \cdot a \cdot c$.

The following propositions are true:

(1) If $a \neq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c = a \cdot (x + \frac{b}{2 \cdot a})^{\mathbf{2}} - \frac{\Delta(a,b,c)}{4 \cdot a}$.

(2) If $a > 0$ and $\Delta(a, b, c) \leq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$.

(3) If $a > 0$ and $\Delta(a, b, c) < 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$.

(4) If $a < 0$ and $\Delta(a, b, c) \leq 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$.

(5) If $a < 0$ and $\Delta(a, b, c) < 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$.

(6) If $a > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a, b, c) \geq 0$.

(7) If $a > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a, b, c) > 0$.

(8) If $a < 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a, b, c) \geq 0$.

(9) If $a < 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a, b, c) > 0$.

(10) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ and $a > 0$, then $\Delta(a, b, c) \leq 0$.

(11) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ and $a < 0$, then $\Delta(a, b, c) \leq 0$.

(12) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ and $a > 0$, then $\Delta(a, b, c) < 0$.

(13) If for every x holds $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ and $a < 0$, then $\Delta(a, b, c) < 0$.

(14) If $a \neq 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$, then $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a, b, c) = 0$.

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(15) Suppose $a \neq 0$ and $\Delta(a,b,c) > 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$. Then $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(16) Suppose $a \neq 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c = a \cdot (x - \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}) \cdot (x - \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a})$.

(17) If $a < 0$ and $\Delta(a,b,c) > 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(18) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} < x$ and $x < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(19) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $x < \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x > \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(20) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ if and only if $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} \leq x$ and $x \leq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(21) Suppose $a < 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ if and only if $x \leq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x \geq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(22) If $a \neq 0$ and $\Delta(a,b,c) = 0$ and $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$, then $x = -\frac{b}{2 \cdot a}$.

(23) If $a > 0$ and $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$.

(24) If $a > 0$ and $\Delta(a,b,c) = 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $x \neq -\frac{b}{2 \cdot a}$.

(25) If $a < 0$ and $(2 \cdot a \cdot x + b)^{\mathbf{2}} - \Delta(a,b,c) > 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$.

(26) If $a < 0$ and $\Delta(a,b,c) = 0$, then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $x \neq -\frac{b}{2 \cdot a}$.

(27) If $a > 0$ and $\Delta(a,b,c) > 0$, then $\frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a} > \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(28) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c < 0$ if and only if $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} < x$ and $x < \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(29) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c > 0$ if and only if $x < \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x > \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(30) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \leq 0$ if and only if $\frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a} \leq x$ and $x \leq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(31) Suppose $a > 0$ and $\Delta(a,b,c) > 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c \geq 0$ if and only if $x \leq \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x \geq \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

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