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Products and Coproducts in Categories

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Summary. A product and coproduct in categories are introduced. The concepts included corresponds to that presented in [7].

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The papers [9], [1], [2], [8], [4], [6], [3], and [5] provide the notation and terminology for this paper.

1. Indexed families

For simplicity we adopt the following rules: I will be a set, x, x_1, x_2, y, y_1, y_2 will be arbitrary, A will be a non-empty set, C, D will be categories, a, b, c, d will be objects of C, and f, g, h, k, p_1, p_2, q_1, q_2, i_1, i_2, j_1, j_2 will be morphisms of C. Let us consider I, x, A, and let F be a function from I into A. Let us assume that $x \in I$. The functor F_x yielding an element of A is defined as follows:

(Def.1) $F_x = F(x)$.

The scheme *LambdaIdx* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_x = \mathcal{F}(x)$

for all values of the parameters.

The following proposition is true

(1) For all functions F_1, F_2 from I into A such that for every x such that $x \in I$ holds $F_{1x} = F_{2x}$ holds $F_1 = F_2$.

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The scheme *FuncIdx_correctn* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

(i) there exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_x = \mathcal{F}(x)$,

(ii) for all functions F_1, F_2 from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $F_{1x} = \mathcal{F}(x)$ and for every x such that $x \in \mathcal{A}$ holds $F_{2x} = \mathcal{F}(x)$ holds $F_1 = F_2$

for all values of the parameters.

Let us consider A, I, and let a be an element of A. Then $I \longmapsto a$ is a function from I into A.

The following proposition is true

(2) For every element a of A such that $x \in I$ holds $(I \longmapsto a)_x = a$.

Let us consider x_1, x_2, y_1, y_2. The functor $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]$ yields a function and is defined as follows:

(Def.2) $[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = (\{x_1\} \longmapsto y_1) +\cdot (\{x_2\} \longmapsto y_2)$.

The following propositions are true:

(3) $\mathrm{dom}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{x_1, x_2\}$ and $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] \subseteq \{y_1, y_2\}$.

(4) If $x_1 \neq x_2$, then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_1) = y_1$ and $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_2) = y_2$.

(5) If $x_1 \neq x_2$, then $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{y_1, y_2\}$.

(6) $[x_1 \longmapsto y, x_2 \longmapsto y] = \{x_1, x_2\} \longmapsto y$.

Let us consider A, x_1, x_2, and let y_1, y_2 be elements of A. Then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]$ is a function from $\{x_1, x_2\}$ into A.

The following proposition is true

(7) If $x_1 \neq x_2$, then for all elements y_1, y_2 of A holds $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]_{x_1} = y_1$ and $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]_{x_2} = y_2$.

2. Indexed families of morphisms

We now define two new functors. Let us consider C, I, and let F be a function from I into the morphisms of C. The functor $\mathrm{dom}_\kappa F(\kappa)$ yielding a function from I into the objects of C is defined as follows:

(Def.3) for every x such that $x \in I$ holds $(\mathrm{dom}_\kappa F(\kappa))_x = \mathrm{dom}(F_x)$.

The functor $\mathrm{cod}_\kappa F(\kappa)$ yielding a function from I into the objects of C is defined by:

(Def.4) for every x such that $x \in I$ holds $(\mathrm{cod}_\kappa F(\kappa))_x = \mathrm{cod}(F_x)$.

We now state four propositions:

(8) $\mathrm{dom}_\kappa(I \longmapsto f)(\kappa) = I \longmapsto \mathrm{dom}\, f$.

(9) $\mathrm{cod}_\kappa(I \longmapsto f)(\kappa) = I \longmapsto \mathrm{cod}\, f$.

(10) $\mathrm{dom}_\kappa[x_1 \longmapsto p_1, x_2 \longmapsto p_2](\kappa) = [x_1 \longmapsto \mathrm{dom}\, p_1, x_2 \longmapsto \mathrm{dom}\, p_2]$.

(11) $\mathrm{cod}_\kappa[x_1 \longmapsto p_1, x_2 \longmapsto p_2](\kappa) = [x_1 \longmapsto \mathrm{cod}\, p_1, x_2 \longmapsto \mathrm{cod}\, p_2]$.

Let us consider C, I, and let F be a function from I into the morphisms of C. The functor F^{op} yields a function from I into the morphisms of C^{op} and is defined as follows:

(Def.5) for every x such that $x \in I$ holds $(F^{\mathrm{op}})_x = (F_x)^{\mathrm{op}}$.

Next we state three propositions:

(12) $(I \longmapsto f)^{\mathrm{op}} = I \longmapsto f^{\mathrm{op}}$.

(13) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]^{\mathrm{op}} = [x_1 \longmapsto {p_1}^{\mathrm{op}}, x_2 \longmapsto {p_2}^{\mathrm{op}}]$.

(14) For every function F from I into the morphisms of C holds $(F^{\mathrm{op}})^{\mathrm{op}} = F$.

Let us consider C, I, and let F be a function from I into the morphisms of C^{op}. The functor ${}^{\mathrm{op}}F$ yielding a function from I into the morphisms of C is defined by:

(Def.6) for every x such that $x \in I$ holds $({}^{\mathrm{op}}F)_x = {}^{\mathrm{op}}(F_x)$.

The following propositions are true:

(15) For every morphism f of C^{op} holds ${}^{\mathrm{op}}(I \longmapsto f) = I \longmapsto {}^{\mathrm{op}}f$.

(16) If $x_1 \neq x_2$, then for all morphisms p_1, p_2 of C^{op} holds ${}^{\mathrm{op}}[x_1 \longmapsto p_1, x_2 \longmapsto p_2] = [x_1 \longmapsto {}^{\mathrm{op}}p_1, x_2 \longmapsto {}^{\mathrm{op}}p_2]$.

(17) For every function F from I into the morphisms of C holds ${}^{\mathrm{op}}(F^{\mathrm{op}}) = F$.

We now define two new functors. Let us consider C, I, and let F be a function from I into the morphisms of C, and let us consider f. The functor $F \cdot f$ yields a function from I into the morphisms of C and is defined as follows:

(Def.7) for every x such that $x \in I$ holds $(F \cdot f)_x = F_x \cdot f$.

The functor $f \cdot F$ yielding a function from I into the morphisms of C is defined by:

(Def.8) for every x such that $x \in I$ holds $(f \cdot F)_x = f \cdot F_x$.

The following four propositions are true:

(18) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2] \cdot f = [x_1 \longmapsto p_1 \cdot f, x_2 \longmapsto p_2 \cdot f]$.

(19) If $x_1 \neq x_2$, then $f \cdot [x_1 \longmapsto p_1, x_2 \longmapsto p_2] = [x_1 \longmapsto f \cdot p_1, x_2 \longmapsto f \cdot p_2]$.

(20) For every function F from I into the morphisms of C such that $\mathrm{dom}_\kappa F(\kappa) = I \longmapsto \mathrm{cod}\, f$ holds $\mathrm{dom}_\kappa F \cdot f(\kappa) = I \longmapsto \mathrm{dom}\, f$ and $\mathrm{cod}_\kappa F \cdot f(\kappa) = \mathrm{cod}_\kappa F(\kappa)$.

(21) For every function F from I into the morphisms of C such that $\mathrm{cod}_\kappa F(\kappa) = I \longmapsto \mathrm{dom}\, f$ holds $\mathrm{dom}_\kappa f \cdot F(\kappa) = \mathrm{dom}_\kappa F(\kappa)$ and $\mathrm{cod}_\kappa f \cdot F(\kappa) = I \longmapsto \mathrm{cod}\, f$.

Let us consider C, I, and let F, G be functions from I into the morphisms of C. The functor $F \cdot G$ yields a function from I into the morphisms of C and is defined by:

(Def.9) for every x such that $x \in I$ holds $(F \cdot G)_x = F_x \cdot G_x$.

We now state four propositions:

(22) For all functions F, G from I into the morphisms of C such that $\mathrm{dom}_\kappa F(\kappa) = \mathrm{cod}_\kappa G(\kappa)$ holds $\mathrm{dom}_\kappa F \cdot G(\kappa) = \mathrm{dom}_\kappa G(\kappa)$ and $\mathrm{cod}_\kappa F \cdot G(\kappa) = \mathrm{cod}_\kappa F(\kappa)$.

(23) If $x_1 \neq x_2$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2] \cdot [x_1 \longmapsto q_1, x_2 \longmapsto q_2] = [x_1 \longmapsto p_1 \cdot q_1, x_2 \longmapsto p_2 \cdot q_2]$.

(24) For every function F from I into the morphisms of C holds $F \cdot f = F \cdot (I \longmapsto f)$.

(25) For every function F from I into the morphisms of C holds $f \cdot F = (I \longmapsto f) \cdot F$.

3. Retractions and coretractions

We now define two new attributes. Let us consider C. A morphism of C is retraction if:

(Def.10) there exists g such that $\mathrm{cod}\, g = \mathrm{dom}\,\mathrm{it}$ and $\mathrm{it} \cdot g = \mathrm{id}_{\mathrm{cod}\,\mathrm{it}}$.

A morphism of C is coretraction if:

(Def.11) there exists g such that $\mathrm{dom}\, g = \mathrm{cod}\,\mathrm{it}$ and $g \cdot \mathrm{it} = \mathrm{id}_{\mathrm{dom}\,\mathrm{it}}$.

The following propositions are true:

(26) If f is retraction, then f is epi.

(27) If f is coretraction, then f is monic.

(28) If f is retraction and g is retraction and $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f$ is retraction.

(29) If f is coretraction and g is coretraction and $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f$ is coretraction.

(30) If $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $g \cdot f$ is retraction, then g is retraction.

(31) If $\mathrm{dom}\, g = \mathrm{cod}\, f$ and $g \cdot f$ is coretraction, then f is coretraction.

(32) If f is retraction and f is monic, then f is invertible.

(33) If f is coretraction and f is epi, then f is invertible.

(34) f is invertible if and only if f is retraction and f is coretraction.

(35) For every functor T from C to D such that f is retraction holds $T(f)$ is retraction.

(36) For every functor T from C to D such that f is coretraction holds $T(f)$ is coretraction.

(37) f is retraction if and only if f^{op} is coretraction.

(38) f is coretraction if and only if f^{op} is retraction.

4. Morphisms determined by a terminal object

Let us consider C, a, b. Let us assume that b is a terminal object. $|_b a$ is a morphism from a to b.

Next we state three propositions:

(39) If b is a terminal object, then $\mathrm{dom}|_b a = a$ and $\mathrm{cod}|_b a = b$.

(40) If b is a terminal object and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$, then $|_b a = f$.

(41) For every morphism f from a to b such that b is a terminal object holds $|_b a = f$.

5. Morphisms determined by an iniatial object

Let us consider C, a, b. Let us assume that a is an initial object. $|^a b$ is a morphism from a to b.

Next we state three propositions:

(42) If a is an initial object, then $\mathrm{dom}|^a b = a$ and $\mathrm{cod}|^a b = b$.

(43) If a is an initial object and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$, then $|^a b = f$.

(44) For every morphism f from a to b such that a is an initial object holds $|^a b = f$.

6. Products

Let us consider C, a, I. A function from I into the morphisms of C is said to be a projections family from a onto I if:

(Def.12) $\mathrm{dom}_\kappa\, \mathrm{it}(\kappa) = I \longmapsto a$.

We now state several propositions:

(45) For every projections family F from a onto I such that $x \in I$ holds $\mathrm{dom}(F_x) = a$.

(46) For every function F from $\emptyset$ into the morphisms of C holds F is a projections family from a onto $\emptyset$.

(47) If $\mathrm{dom}\, f = a$, then $\{y\} \longmapsto f$ is a projections family from a onto $\{y\}$.

(48) If $\mathrm{dom}\, p_1 = a$ and $\mathrm{dom}\, p_2 = a$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$ is a projections family from a onto $\{x_1, x_2\}$.

(49) For every projections family F from a onto $\emptyset$ holds $F = \square$.

(50) For every projections family F from a onto I such that $\mathrm{cod}\, f = a$ holds $F \cdot f$ is a projections family from $\mathrm{dom}\, f$ onto I.

(51) For every function F from I into the morphisms of C and for every projections family G from a onto I such that $\mathrm{dom}_\kappa\, F(\kappa) = \mathrm{cod}_\kappa\, G(\kappa)$ holds $F \cdot G$ is a projections family from a onto I.

(52) For every projections family F from $\mathrm{cod}\, f$ onto I holds $f^{\mathrm{op}} \cdot F^{\mathrm{op}} = (F \cdot f)^{\mathrm{op}}$.

Let us consider C, a, I, and let F be a function from I into the morphisms of C. We say that a is a product w.r.t. F if and only if the conditions (Def.13) is satisfied.

(Def.13) (i) F is a projections family from a onto I,

(ii) for every b and for every projections family F' from b onto I such that $\mathrm{cod}_\kappa F(\kappa) = \mathrm{cod}_\kappa F'(\kappa)$ there exists h such that $h \in \hom(b, a)$ and for every k such that $k \in \hom(b, a)$ holds for every x such that $x \in I$ holds $F_x \cdot k = F'_x$ if and only if $h = k$.

One can prove the following propositions:

(53) For every projections family F from c onto I and for every projections family F' from d onto I such that c is a product w.r.t. F and d is a product w.r.t. F' and $\mathrm{cod}_\kappa F(\kappa) = \mathrm{cod}_\kappa F'(\kappa)$ holds c and d are isomorphic.

(54) For every projections family F from c onto I such that c is a product w.r.t. F and for all x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ holds $\hom(\mathrm{cod}(F_{x_1}), \mathrm{cod}(F_{x_2})) \neq \emptyset$ and for every x such that $x \in I$ holds F_x is retraction.

(55) For every function F from $\emptyset$ into the morphisms of C holds a is a product w.r.t. F if and only if a is a terminal object.

(56) For every projections family F from a onto I such that a is a product w.r.t. F and $\mathrm{dom}\, f = b$ and $\mathrm{cod}\, f = a$ and f is invertible holds b is a product w.r.t. $F \cdot f$.

(57) a is a product w.r.t. $\{y\} \longmapsto \mathrm{id}_a$.

(58) For every projections family F from a onto I such that a is a product w.r.t. F and for every x such that $x \in I$ holds $\mathrm{cod}(F_x)$ is a terminal object holds a is a terminal object.

Let us consider C, c, p_1, p_2. We say that c is a product w.r.t. p_1 and p_2 if and only if the conditions (Def.14) is satisfied.

(Def.14) (i) $\mathrm{dom}\, p_1 = c$,

(ii) $\mathrm{dom}\, p_2 = c$,

(iii) for all d, f, g such that $f \in \hom(d, \mathrm{cod}\, p_1)$ and $g \in \hom(d, \mathrm{cod}\, p_2)$ there exists h such that $h \in \hom(d, c)$ and for every k such that $k \in \hom(d, c)$ holds $p_1 \cdot k = f$ and $p_2 \cdot k = g$ if and only if $h = k$.

The following propositions are true:

(59) If $x_1 \neq x_2$, then c is a product w.r.t. p_1 and p_2 if and only if c is a product w.r.t. $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$.

(60) Suppose $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$. Let p_1 be a morphism from c to a. Let p_2 be a morphism from c to b. Then c is a product w.r.t. p_1 and p_2 if and only if for every d such that $\hom(d, a) \neq \emptyset$ and $\hom(d, b) \neq \emptyset$ holds $\hom(d, c) \neq \emptyset$ and for every morphism f from d to a and for every morphism g from d to b there exists a morphism h from d to c such that for every morphism k from d to c holds $p_1 \cdot k = f$ and $p_2 \cdot k = g$ if and only if $h = k$.

(61) If c is a product w.r.t. p_1 and p_2 and d is a product w.r.t. q_1 and q_2 and $\mathrm{cod}\, p_1 = \mathrm{cod}\, q_1$ and $\mathrm{cod}\, p_2 = \mathrm{cod}\, q_2$, then c and d are isomorphic.

(62) If c is a product w.r.t. p_1 and p_2 and $\hom(\operatorname{cod} p_1, \operatorname{cod} p_2) \neq \emptyset$ and $\hom(\operatorname{cod} p_2, \operatorname{cod} p_1) \neq \emptyset$, then p_1 is retraction and p_2 is retraction.

(63) If c is a product w.r.t. p_1 and p_2 and $h \in \hom(c, c)$ and $p_1 \cdot h = p_1$ and $p_2 \cdot h = p_2$, then $h = \mathrm{id}_c$.

(64) If c is a product w.r.t. p_1 and p_2 and $\operatorname{dom} f = d$ and $\operatorname{cod} f = c$ and f is invertible, then d is a product w.r.t. $p_1 \cdot f$ and $p_2 \cdot f$.

(65) If c is a product w.r.t. p_1 and p_2 and $\operatorname{cod} p_2$ is a terminal object, then c and $\operatorname{cod} p_1$ are isomorphic.

(66) If c is a product w.r.t. p_1 and p_2 and $\operatorname{cod} p_1$ is a terminal object, then c and $\operatorname{cod} p_2$ are isomorphic.

7. Coproducts

Let us consider C, c, I. A function from I into the morphisms of C is said to be a injections family into c on I if:

(Def.15) $\operatorname{cod}_\kappa \operatorname{it}(\kappa) = I \longmapsto c$.

We now state a number of propositions:

(67) For every injections family F into c on I such that $x \in I$ holds $\operatorname{cod}(F_x) = c$.

(68) For every function F from $\emptyset$ into the morphisms of C holds F is a injections family into a on $\emptyset$.

(69) If $\operatorname{cod} f = a$, then $\{y\} \longmapsto f$ is a injections family into a on $\{y\}$.

(70) If $\operatorname{cod} p_1 = c$ and $\operatorname{cod} p_2 = c$, then $[x_1 \longmapsto p_1, x_2 \longmapsto p_2]$ is a injections family into c on $\{x_1, x_2\}$.

(71) For every injections family F into c on $\emptyset$ holds $F = \square$.

(72) For every injections family F into b on I such that $\operatorname{dom} f = b$ holds $f \cdot F$ is a injections family into $\operatorname{cod} f$ on I.

(73) For every injections family F into b on I and for every function G from I into the morphisms of C such that $\operatorname{dom}_\kappa F(\kappa) = \operatorname{cod}_\kappa G(\kappa)$ holds $F \cdot G$ is a injections family into b on I.

(74) For every function F from I into the morphisms of C holds F is a projections family from c onto I if and only if F^{op} is a injections family into c^{op} on I.

(75) For every function F from I into the morphisms of C^{op} and for every object c of C^{op} holds F is a injections family into c on I if and only if ${}^{\mathrm{op}}F$ is a projections family from ${}^{\mathrm{op}}c$ onto I.

(76) For every injections family F into $\operatorname{dom} f$ on I holds $F^{\mathrm{op}} \cdot f^{\mathrm{op}} = (f \cdot F)^{\mathrm{op}}$.

Let us consider C, c, I, and let F be a function from I into the morphisms of C. We say that c is a coproduct w.r.t. F if and only if the conditions (Def.16) is satisfied.

(Def.16) (i) F is a injections family into c on I,

(ii) for every d and for every injections family F' into d on I such that $\mathrm{dom}_\kappa F(\kappa) = \mathrm{dom}_\kappa F'(\kappa)$ there exists h such that $h \in \mathrm{hom}(c, d)$ and for every k such that $k \in \mathrm{hom}(c, d)$ holds for every x such that $x \in I$ holds $k \cdot F_x = F'_x$ if and only if $h = k$.

One can prove the following propositions:

(77) For every function F from I into the morphisms of C holds c is a product w.r.t. F if and only if c^{op} is a coproduct w.r.t. F^{op}.

(78) For every injections family F into c on I and for every injections family F' into d on I such that c is a coproduct w.r.t. F and d is a coproduct w.r.t. F' and $\mathrm{dom}_\kappa F(\kappa) = \mathrm{dom}_\kappa F'(\kappa)$ holds c and d are isomorphic.

(79) For every injections family F into c on I such that c is a coproduct w.r.t. F and for all x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ holds $\mathrm{hom}(\mathrm{dom}(F_{x_1}), \mathrm{dom}(F_{x_2})) \neq \emptyset$ and for every x such that $x \in I$ holds F_x is coretraction.

(80) For every injections family F into a on I such that a is a coproduct w.r.t. F and $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b$ and f is invertible holds b is a coproduct w.r.t. $f \cdot F$.

(81) For every injections family F into a on $\emptyset$ holds a is a coproduct w.r.t. F if and only if a is an initial object.

(82) a is a coproduct w.r.t. $\{y\} \longmapsto \mathrm{id}_a$.

(83) For every injections family F into a on I such that a is a coproduct w.r.t. F and for every x such that $x \in I$ holds $\mathrm{dom}(F_x)$ is an initial object holds a is an initial object.

Let us consider C, c, i_1, i_2. We say that c is a coproduct w.r.t. i_1 and i_2 if and only if the conditions (Def.17) is satisfied.

(Def.17) (i) $\mathrm{cod}\, i_1 = c$,

(ii) $\mathrm{cod}\, i_2 = c$,

(iii) for all d, f, g such that $f \in \mathrm{hom}(\mathrm{dom}\, i_1, d)$ and $g \in \mathrm{hom}(\mathrm{dom}\, i_2, d)$ there exists h such that $h \in \mathrm{hom}(c, d)$ and for every k such that $k \in \mathrm{hom}(c, d)$ holds $k \cdot i_1 = f$ and $k \cdot i_2 = g$ if and only if $h = k$.

We now state several propositions:

(84) c is a product w.r.t. p_1 and p_2 if and only if c^{op} is a coproduct w.r.t. ${p_1}^{\mathrm{op}}$ and ${p_2}^{\mathrm{op}}$.

(85) If $x_1 \neq x_2$, then c is a coproduct w.r.t. i_1 and i_2 if and only if c is a coproduct w.r.t. $[x_1 \longmapsto i_1, x_2 \longmapsto i_2]$.

(86) If c is a coproduct w.r.t. i_1 and i_2 and d is a coproduct w.r.t. j_1 and j_2 and $\mathrm{dom}\, i_1 = \mathrm{dom}\, j_1$ and $\mathrm{dom}\, i_2 = \mathrm{dom}\, j_2$, then c and d are isomorphic.

(87) Suppose $\mathrm{hom}(a, c) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$. Let i_1 be a morphism from a to c. Let i_2 be a morphism from b to c. Then c is a coproduct w.r.t. i_1 and i_2 if and only if for every d such that $\mathrm{hom}(a, d) \neq \emptyset$ and $\mathrm{hom}(b, d) \neq \emptyset$ holds $\mathrm{hom}(c, d) \neq \emptyset$ and for every morphism f from a to d and for every

morphism g from b to d there exists a morphism h from c to d such that for every morphism k from c to d holds $k \cdot i_1 = f$ and $k \cdot i_2 = g$ if and only if $h = k$.

(88) If c is a coproduct w.r.t. i_1 and i_2 and $\hom(\operatorname{dom} i_1, \operatorname{dom} i_2) \neq \emptyset$ and $\hom(\operatorname{dom} i_2, \operatorname{dom} i_1) \neq \emptyset$, then i_1 is coretraction and i_2 is coretraction.

(89) If c is a coproduct w.r.t. i_1 and i_2 and $h \in \hom(c, c)$ and $h \cdot i_1 = i_1$ and $h \cdot i_2 = i_2$, then $h = \mathrm{id}_c$.

(90) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} f = c$ and $\operatorname{cod} f = d$ and f is invertible, then d is a coproduct w.r.t. $f \cdot i_1$ and $f \cdot i_2$.

(91) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} i_2$ is an initial object, then $\operatorname{dom} i_1$ and c are isomorphic.

(92) If c is a coproduct w.r.t. i_1 and i_2 and $\operatorname{dom} i_1$ is an initial object, then $\operatorname{dom} i_2$ and c are isomorphic.

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