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Sequences in Metric Spaces

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Summary. Sequences in metric spaces are defined. The article contains definitions of bounded, convergent, Cauchy sequences. The subsequences are introduced too. Some theorems concerning sequences are proved.

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The terminology and notation used in this paper have been introduced in the following articles: [11], [14], [4], [5], [3], [6], [13], [12], [7], [10], [8], [9], [1], and [2]. For simplicity we follow a convention: X will be a metric space, x, y, z will be elements of the carrier of X, V will be a subset of the carrier of X, A will be a non-empty set, a will be an element of A, G will be a function from $[\!: A, A :\!]$ into $\mathbb{R}$, k, n, m will be natural numbers, and r will be a real number. The following propositions are true:

(1) $|\rho(x, z) - \rho(y, z)| \leq \rho(x, y)$.

(2) If G is a metric of A, then for all elements a, b of A holds $0 \leq G(a, b)$.

Let us consider A, G. We say that G is not a pseudo metric if and only if:

(Def.1) for all elements a, b of A holds $G(a, b) = 0$ if and only if $a = b$.

Let us consider A, G. We say that G is symmetric if and only if:

(Def.2) for all elements a, b of A holds $G(a, b) = G(b, a)$.

Let us consider A, G. We say that G satisfies triangle inequality if and only if:

(Def.3) for all elements a, b, c of A holds $G(a, c) \leq G(a, b) + G(b, c)$.

Next we state three propositions:

(3) G is a metric of A if and only if G is not a pseudo metric and G is symmetric and G satisfies triangle inequality.

(4) For every strict metric space X holds the distance of X is not a pseudo metric and the distance of X is symmetric and the distance of X satisfies triangle inequality.

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(5) G is a metric of A if and only if G is not a pseudo metric and for all elements a, b, c of A holds $G(b, c) \leq G(a, b) + G(a, c)$.

Let us consider A, G. Let us assume that G is a metric of A. The functor $\widetilde{G}_A$ yielding a function from $[\!: A, A :\!]$ into $\mathbb{R}$ is defined as follows:

(Def.4) for all elements a, b of A holds $\widetilde{G}_A(a, b) = \frac{G(a, b)}{1+G(a, b)}$.

The following proposition is true

(6) If G is a metric of A, then $\widetilde{G}_A$ is a metric of A.

Let X be a metric space. A sequence of elements of X is defined by:

(Def.5) it is a function from $\mathbb{N}$ into the carrier of X.

Let X be a metric space. We see that the sequence of elements of X is a function from $\mathbb{N}$ into the carrier of X.

Next we state the proposition

(7) For every function F from $\mathbb{N}$ into the carrier of X holds F is a sequence of elements of X.

We follow the rules: S, S_1, T denote sequences of elements of X, N_1 denotes an increasing sequence of naturals, and F denotes a function from $\mathbb{N}$ into the carrier of X. The following propositions are true:

(8) F is a sequence of elements of X if and only if for every a such that $a \in \mathbb{N}$ holds $F(a)$ is an element of the carrier of X.

(9) For all S, T such that for every n holds $S(n) = T(n)$ holds $S = T$.

(10) For every x there exists S such that $\operatorname{rng} S = \{x\}$.

(11) If there exists x such that for every n holds $S(n) = x$, then there exists x such that $\operatorname{rng} S = \{x\}$.

Let us consider X, S. We say that S is constant if and only if:

(Def.6) there exists x such that for every n holds $S(n) = x$.

The following proposition is true

(13)[1] S is constant if and only if there exists x such that $\operatorname{rng} S = \{x\}$.

Let us consider X, S. We say that S is convergent if and only if:

(Def.7) there exists x such that for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

Let us consider X, S, x. We say that S is convergent to x if and only if:

(Def.8) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

Let us consider X, S. We say that S satisfies the Cauchy condition if and only if:

(Def.9) for every r such that $0 < r$ there exists m such that for all n, k such that $m \leq n$ and $m \leq k$ holds $\rho(S(n), S(k)) < r$.

Let us consider X, V. We say that V is bounded if and only if:

[1]The proposition (12) has been removed.

(Def.10) there exist r, x such that $0 < r$ and $V \subseteq \mathrm{Ball}(x, r)$.

Let us consider X, S. We say that S is bounded if and only if:

(Def.11) there exist r, x such that $0 < r$ and $\mathrm{rng}\, S \subseteq \mathrm{Ball}(x, r)$.

Let us consider X, V, S. We say that V contains almost all sequence S if and only if:

(Def.12) there exists m such that for every n such that $m \leq n$ holds $S(n) \in V$.

Let us consider X, s_1, s_2. We say that s_1 is a subsequence of s_2 if and only if:

(Def.13) there exists N_1 such that $s_1 = s_2 \cdot N_1$.

Next we state the proposition

(16)[2] S is convergent to x if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

We now state three propositions:

(20)[3] S is bounded if and only if there exist r, x such that $0 < r$ and for every n holds $S(n) \in \mathrm{Ball}(x, r)$.

(21) If S is convergent to x, then S is convergent.

(22) If S is convergent, then there exists x such that S is convergent to x.

Let us consider X, S, x. The functor $\rho(S, x)$ yields a sequence of real numbers and is defined as follows:

(Def.14) for every n holds $(\rho(S, x))(n) = \rho(S(n), x)$.

Next we state the proposition

(23) $\rho(S, x)$ is a sequence of real numbers if and only if for every n holds $(\rho(S, x))(n) = \rho(S(n), x)$.

Let us consider X, S, T. The functor $\rho(S, T)$ yields a sequence of real numbers and is defined by:

(Def.15) for every n holds $(\rho(S, T))(n) = \rho(S(n), T(n))$.

Next we state the proposition

(24) $\rho(S, T)$ is a sequence of real numbers if and only if for every n holds $(\rho(S, T))(n) = \rho(S(n), T(n))$.

Let us consider X, S. Let us assume that S is convergent. The functor $\lim S$ yields an element of the carrier of X and is defined as follows:

(Def.16) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), \lim S) < r$.

One can prove the following propositions:

(25) If S is convergent, then $\lim S = x$ if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

[2]The propositions (14) and (15) have been removed.

[3]The propositions (17)–(19) have been removed.

(26) If S is convergent to x, then $\lim S = x$.

(27) S is convergent to x if and only if S is convergent and $\lim S = x$.

(28) If S is convergent, then there exists x such that S is convergent to x and $\lim S = x$.

(29) S is convergent to x if and only if $\rho(S, x)$ is convergent and $\lim \rho(S, x) = 0$.

(30) If S is convergent to x, then for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S.

(31) If for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S, then for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(32) If for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S, then S is convergent to x.

(33) S is convergent to x if and only if for every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S.

(34) S is convergent to x if and only if for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(35) For every r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ contains almost all sequence S if and only if for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(36) If S is convergent and T is convergent, then $\rho(\lim S, \lim T) = \lim \rho(S, T)$.

(37) If S is convergent to x and S is convergent to y, then $x = y$.

(38) If S is constant, then S is convergent.

(39) If S is convergent to x and S_1 is a subsequence of S, then S_1 is convergent to x.

(40) If S satisfies the Cauchy condition and S_1 is a subsequence of S, then S_1 satisfies the Cauchy condition.

(41) If S is convergent, then S satisfies the Cauchy condition.

(42) If S is constant, then S satisfies the Cauchy condition.

(43) If S is convergent, then S is bounded.

(44) If S satisfies the Cauchy condition, then S is bounded.

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