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Category of Left Modules

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Summary. We define the category of left modules over an associative ring. The carriers of the modules are included in a universum. The universum is a parameter of the category.

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The papers [12], [1], [2], [4], [5], [7], [3], [11], [10], [9], [6], and [8] provide the terminology and notation for this paper. For simplicity we adopt the following convention: x, y are arbitrary, D is a non-empty set, U_1 is a universal class, R is an associative ring, and G, H are left modules over R. Let us consider R. A non-empty set is said to be a non-empty set of left-modules of R if:

(Def.1) for every element x of it holds x is a strict left module over R.

In the sequel V is a non-empty set of left-modules of R. Let us consider R, V. We see that the element of V is a left module over R.

We now state two propositions:

(1) For every left module morphism f of R and for every element x of $\{f\}$ holds x is a left module morphism of R.

(2) For every strict morphism f from G to H and for every element x of $\{f\}$ holds x is a strict morphism from G to H.

Let us consider R. A non-empty set is said to be a non-empty set of morphisms of left-modules of R if:

(Def.2) for every element x of it holds x is a strict left module morphism of R.

Let us consider R, and let M be a non-empty set of morphisms of left-modules of R. We see that the element of M is a left module morphism of R.

Next we state the proposition

(3) For every strict left module morphism f of R holds $\{f\}$ is a non-empty set of morphisms of left-modules of R.

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Let us consider R, G, H. A non-empty set of morphisms of left-modules of R is called a non-empty set of morphisms of left-modules from G into H if:

(Def.3) for every element x of it holds x is a strict morphism from G to H.

The following two propositions are true:

(4) D is a non-empty set of morphisms of left-modules from G into H if and only if for every element x of D holds x is a strict morphism from G to H.

(5) For every strict morphism f from G to H holds $\{f\}$ is a non-empty set of morphisms of left-modules from G into H.

Let us consider R, G, H. The functor $\mathrm{Morphs}(G,H)$ yields a non-empty set of morphisms of left-modules from G into H and is defined as follows:

(Def.4) $x \in \mathrm{Morphs}(G,H)$ if and only if x is a strict morphism from G to H.

Let us consider R, G, H, and let M be a non-empty set of morphisms of left-modules from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y, R. The predicate $\mathrm{P_{ob}}\, x, y, R$ is defined by:

(Def.5) there exist arbitrary x_1, x_2 such that $x = \langle x_1, x_2\rangle$ and there exists a strict left module G over R such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the left multiplication of G.

One can prove the following propositions:

(6) For arbitrary x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1, R$ and $\mathrm{P_{ob}}\, x, y_2, R$ holds $y_1 = y_2$.

(7) For every U_1 there exists x such that $x \in \{\langle G, f\rangle\}$, where G ranges over elements of $\mathrm{GroupObj}(U_1)$, and f ranges over elements of $\{\emptyset\}^{[:\text{the carrier of } R, \{\emptyset\}:]}$ and $\mathrm{P_{ob}}\, x, {}_R\Theta, R$.

Let us consider U_1, R. The functor $\mathrm{LModObj}(U_1,R)$ yielding a non-empty set is defined as follows:

(Def.6) for every y holds $y \in \mathrm{LModObj}(U_1,R)$ if and only if there exists x such that $x \in \{\langle G, f\rangle\}$, where G ranges over elements of $\mathrm{GroupObj}(U_1)$, and f ranges over elements of $\{\emptyset\}^{[:\text{the carrier of } R, \{\emptyset\}:]}$ and $\mathrm{P_{ob}}\, x, y, R$.

One can prove the following two propositions:

(8) ${}_R\Theta \in \mathrm{LModObj}(U_1,R)$.

(9) For every element x of $\mathrm{LModObj}(U_1,R)$ holds x is a strict left module over R.

Let us consider U_1, R. Then $\mathrm{LModObj}(U_1,R)$ is a non-empty set of left-modules of R.

Let us consider R, V. The functor $\mathrm{Morphs}\, V$ yields a non-empty set of morphisms of left-modules of R and is defined as follows:

(Def.7) for every x holds $x \in \mathrm{Morphs}\, V$ if and only if there exist strict elements G, H of V such that x is a strict morphism from G to H.

We now define two new functors. Let us consider R, V, and let F be an element of $\operatorname{Morphs} V$. The functor $\operatorname{dom}' F$ yields an element of V and is defined as follows:

(Def.8) $\operatorname{dom}' F = \operatorname{dom} F$.

The functor $\operatorname{cod}' F$ yields an element of V and is defined by:

(Def.9) $\operatorname{cod}' F = \operatorname{cod} F$.

Let us consider R, V, and let G be an element of V. The functor I_G yielding a strict element of $\operatorname{Morphs} V$ is defined as follows:

(Def.10) $\mathrm{I}_G = \mathrm{I}_G$.

We now define three new functors. Let us consider R, V. The functor $\operatorname{dom} V$ yields a function from $\operatorname{Morphs} V$ into V and is defined by:

(Def.11) for every element f of $\operatorname{Morphs} V$ holds $(\operatorname{dom} V)(f) = \operatorname{dom}' f$.

The functor $\operatorname{cod} V$ yields a function from $\operatorname{Morphs} V$ into V and is defined as follows:

(Def.12) for every element f of $\operatorname{Morphs} V$ holds $(\operatorname{cod} V)(f) = \operatorname{cod}' f$.

The functor I_V yields a function from V into $\operatorname{Morphs} V$ and is defined by:

(Def.13) for every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

One can prove the following three propositions:

(10) For all elements g, f of $\operatorname{Morphs} V$ such that $\operatorname{dom}' g = \operatorname{cod}' f$ there exist strict elements G_1, G_2, G_3 of V such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(11) For all elements g, f of $\operatorname{Morphs} V$ such that $\operatorname{dom}' g = \operatorname{cod}' f$ holds $g \cdot f \in \operatorname{Morphs} V$.

(12) For all elements g, f of $\operatorname{Morphs} V$ such that $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f \in \operatorname{Morphs} V$.

Let us consider R, V. The functor $\operatorname{comp} V$ yields a partial function from $[\!:\operatorname{Morphs} V, \operatorname{Morphs} V:\!]$ to $\operatorname{Morphs} V$ and is defined by:

(Def.14) for all elements g, f of $\operatorname{Morphs} V$ holds $\langle g, f\rangle \in \operatorname{dom} \operatorname{comp} V$ if and only if $\operatorname{dom}' g = \operatorname{cod}' f$ and for all elements g, f of $\operatorname{Morphs} V$ such that $\langle g, f\rangle \in \operatorname{dom} \operatorname{comp} V$ holds $(\operatorname{comp} V)(\langle g, f\rangle) = g \cdot f$.

The following proposition is true

(13) For all elements g, f of $\operatorname{Morphs} V$ holds $\langle g, f\rangle \in \operatorname{dom} \operatorname{comp} V$ if and only if $\operatorname{dom} g = \operatorname{cod} f$.

Let us consider U_1, R. The functor $\operatorname{LModCat}(U_1, R)$ yields a strict category structure and is defined by:

(Def.15) $\operatorname{LModCat}(U_1, R) = \langle \operatorname{LModObj}(U_1, R), \operatorname{Morphs} \operatorname{LModObj}(U_1, R), \operatorname{dom} \operatorname{LModObj}(U_1, R), \operatorname{cod} \operatorname{LModObj}(U_1, R), \operatorname{comp} \operatorname{LModObj}(U_1, R), \mathrm{I}_{\operatorname{LModObj}(U_1, R)}\rangle$.

One can prove the following propositions:

(14) For all morphisms f, g of $\operatorname{LModCat}(U_1, R)$ holds $\langle g, f\rangle \in \operatorname{dom}$ (the composition of $\operatorname{LModCat}(U_1, R)$) if and only if $\operatorname{dom} g = \operatorname{cod} f$.

(15) Let f be a morphism of $\mathrm{LModCat}(U_1, R)$. Then for every element f' of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ and for every object b of $\mathrm{LModCat}(U_1, R)$ and for every element b' of $\mathrm{LModObj}(U_1, R)$ holds f is a strict element of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ and f' is a morphism of $\mathrm{LModCat}(U_1, R)$ and b is a strict element of $\mathrm{LModObj}(U_1, R)$ and b' is an object of $\mathrm{LModCat}(U_1, R)$.

(16) For every object b of $\mathrm{LModCat}(U_1, R)$ and for every element b' of $\mathrm{LModObj}(U_1, R)$ such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(17) For every morphism f of $\mathrm{LModCat}(U_1, R)$ and for every element f' of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$ such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(18) Let f, g be morphisms of $\mathrm{LModCat}(U_1, R)$. Let f', g' be elements of $\mathrm{Morphs}\,\mathrm{LModObj}(U_1, R)$. Suppose $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g', f' \rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{LModObj}(U_1, R)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ if and only if $\mathrm{dom}\, f' = \mathrm{dom}\, g'$,

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ if and only if $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1, R. Then $\mathrm{LModCat}(U_1, R)$ is a strict category.

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