FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Category of Rings

Michał Muzalewski
Warsaw University
Białystok

Summary. We define the category of non-associative rings. The carriers of the rings are included in a universum. The universum is a parameter of the category.

MML Identifier: `RINGCAT1`.

The papers [14], [2], [15], [3], [1], [12], [7], [8], [5], [4], [13], [11], [6], [10], and [9] provide the terminology and notation for this paper. For simplicity we follow a convention: x, y will be arbitrary, D will be a non-empty set, U_1 will be a universal class, and G, H will be field structures. Let us consider G, H. A map from G into H is a function from the carrier of G into the carrier of H.

Let G_1, G_2, G_3 be field structures, and let f be a map from G_1 into G_2, and let g be a map from G_2 into G_3. Then $g \cdot f$ is a map from G_1 into G_3.

Let us consider G. The functor id_G yields a map from G into G and is defined by:

(Def.1) $\mathrm{id}_G = \mathrm{id}_{(\text{the carrier of } G)}$.

The following propositions are true:

(1) For every scalar x of G holds $\mathrm{id}_G(x) = x$.

(2) For every map f from G into H holds $f \cdot \mathrm{id}_G = f$ and $\mathrm{id}_H \cdot f = f$.

Let us consider G, H. A map from G into H is linear if:

(Def.2) for all scalars x, y of G holds $\mathrm{it}(x + y) = \mathrm{it}(x) + \mathrm{it}(y)$ and for all scalars x, y of G holds $\mathrm{it}(x \cdot y) = \mathrm{it}(x) \cdot \mathrm{it}(y)$ and $\mathrm{it}(1_G) = 1_H$.

We now state the proposition

(3) For all G_1, G_2, G_3 being field structures and for every map f from G_1 into G_2 and for every map g from G_2 into G_3 such that f is linear and g is linear holds $g \cdot f$ is linear.

ISSN 0777–4028

We consider ring morphisms structures which are systems

⟨a dom-map, a cod-map, a `Fun`⟩,

where the dom-map, the cod-map are a ring and the `Fun` is a map from the dom-map into the cod-map.

We now define three new functors. Let us consider f. The functor $\mathrm{dom}\, f$ yields a ring and is defined by:

(Def.3) $\mathrm{dom}\, f =$ the dom-map of f.

The functor $\mathrm{cod}\, f$ yields a ring and is defined by:

(Def.4) $\mathrm{cod}\, f =$ the cod-map of f.

The functor $\mathrm{fun}\, f$ yields a map from the dom-map of f into the cod-map of f and is defined by:

(Def.5) $\mathrm{fun}\, f =$ the `Fun` of f.

In the sequel G, H, G_1, G_2, G_3, G_4 will denote rings. A ring morphisms structure is called a morphism of rings if:

(Def.6) fun it is linear.

Let us consider G. The functor I_G yields a strict morphism of rings and is defined as follows:

(Def.7) $\mathrm{I}_G = \langle G, G, \mathrm{id}_G \rangle$.

Let us consider G, H. The predicate $G \leq H$ is defined as follows:

(Def.8) there exists a morphism F of rings such that $\mathrm{dom}\, F = G$ and $\mathrm{cod}\, F = H$.

We now state the proposition

(4) $G \leq G$.

Let us consider G, H. Let us assume that $G \leq H$. A strict morphism of rings is said to be a morphism from G to H if:

(Def.9) dom it $= G$ and cod it $= H$.

Let us consider G. Then I_G is a strict morphism from G to G.

We now state three propositions:

(5) For all morphisms g, f of rings such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ there exist G_1, G_2, G_3 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and the ring morphisms structure of g is a morphism from G_2 to G_3 and the ring morphisms structure of f is a morphism from G_1 to G_2.

(6) For every strict morphism F of rings holds F is a morphism from $\mathrm{dom}\, F$ to $\mathrm{cod}\, F$ and $\mathrm{dom}\, F \leq \mathrm{cod}\, F$.

(7) For every strict morphism F of rings there exist G, H and there exists a map f from G into H such that F is a morphism from G to H and $F = \langle G, H, f \rangle$ and f is linear.

Let G, F be morphisms of rings. Let us assume that $\mathrm{dom}\, G = \mathrm{cod}\, F$. The functor $G \cdot F$ yields a strict morphism of rings and is defined by:

(Def.10) for all G_1, G_2, G_3 and for every map g from G_2 into G_3 and for every map f from G_1 into G_2 such that the ring morphisms structure of $G = \langle G_2, G_3, g\rangle$ and the ring morphisms structure of $F = \langle G_1, G_2, f\rangle$ holds $G \cdot F = \langle G_1, G_3, g \cdot f\rangle$.

We now state two propositions:

(8) If $G_1 \leq G_2$ and $G_2 \leq G_3$, then $G_1 \leq G_3$.

(9) For every morphism G from G_2 to G_3 and for every morphism F from G_1 to G_2 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ holds $G \cdot F$ is a morphism from G_1 to G_3.

Let us consider G_1, G_2, G_3, and let G be a morphism from G_2 to G_3, and let F be a morphism from G_1 to G_2. Let us assume that $G_1 \leq G_2$ and $G_2 \leq G_3$. The functor $F[G]$ yields a strict morphism from G_1 to G_3 and is defined as follows:

(Def.11) $F[G] = G \cdot F$.

The following propositions are true:

(10) For all strict morphisms f, g of rings such that $\operatorname{dom} g = \operatorname{cod} f$ there exist G_1, G_2, G_3 and there exists a map f_0 from G_1 into G_2 and there exists a map g_0 from G_2 into G_3 such that $f = \langle G_1, G_2, f_0\rangle$ and $g = \langle G_2, G_3, g_0\rangle$ and $g \cdot f = \langle G_1, G_3, g_0 \cdot f_0\rangle$.

(11) For all strict morphisms f, g of rings such that $\operatorname{dom} g = \operatorname{cod} f$ holds $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$ and $\operatorname{cod}(g \cdot f) = \operatorname{cod} g$.

(12) For every morphism f from G_1 to G_2 and for every morphism g from G_2 to G_3 and for every morphism h from G_3 to G_4 such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and $G_3 \leq G_4$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(13) For all strict morphisms f, g, h of rings such that $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(14) $\operatorname{dom}(\mathrm{I}_G) = G$ and $\operatorname{cod}(\mathrm{I}_G) = G$ and for every strict morphism f of rings such that $\operatorname{cod} f = G$ holds $\mathrm{I}_G \cdot f = f$ and for every strict morphism g of rings such that $\operatorname{dom} g = G$ holds $g \cdot \mathrm{I}_G = g$.

A non-empty set is said to be a non-empty set of rings if:

(Def.12) for every element x of it holds x is a strict ring.

In the sequel V denotes a non-empty set of rings. Let us consider V. We see that the element of V is a ring.

One can prove the following two propositions:

(15) For every strict morphism f of rings and for every element x of $\{f\}$ holds x is a strict morphism of rings.

(16) For every morphism f from G to H and for every element x of $\{f\}$ holds x is a morphism from G to H.

A non-empty set is said to be a non-empty set of morphisms of rings if:

(Def.13) for every element x of it holds x is a strict morphism of rings.

Let M be a non-empty set of morphisms of rings. We see that the element of M is a morphism of rings.

Next we state the proposition

(17) For every strict morphism f of rings holds $\{f\}$ is a non-empty set of morphisms of rings.

Let us consider G, H. A non-empty set of morphisms of rings is called a non-empty set of morphisms from G into H if:

(Def.14) for every element x of it holds x is a morphism from G to H.

The following two propositions are true:

(18) D is a non-empty set of morphisms from G into H if and only if for every element x of D holds x is a morphism from G to H.

(19) For every morphism f from G to H holds $\{f\}$ is a non-empty set of morphisms from G into H.

Let us consider G, H. Let us assume that $G \leq H$. The functor $\mathrm{Morphs}(G, H)$ yielding a non-empty set of morphisms from G into H is defined by:

(Def.15) $x \in \mathrm{Morphs}(G, H)$ if and only if x is a morphism from G to H.

Let us consider G, H, and let M be a non-empty set of morphisms from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y. The predicate $\mathrm{P_{ob}}\, x, y$ is defined by the condition (Def.16).

(Def.16) There exist arbitrary x_1, x_2, x_3, x_4, x_5, x_6 such that $x = \langle\langle x_1, x_2, x_3, x_4\rangle$, x_5, $x_6\rangle$ and there exists a strict ring G such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the addition of G and $x_3 =$ the reverse-map of G and $x_4 =$ the zero of G and $x_5 =$ the multiplication of G and $x_6 =$ the unity of G.

We now state two propositions:

(20) For arbitrary x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1$ and $\mathrm{P_{ob}}\, x, y_2$ holds $y_1 = y_2$.

(21) There exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, \mathrm{Z}_3$.

Let us consider U_1. The functor $\mathrm{RingObj}(U_1)$ yielding a non-empty set is defined as follows:

(Def.17) for every y holds $y \in \mathrm{RingObj}(U_1)$ if and only if there exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, y$.

We now state two propositions:

(22) $\mathrm{Z}_3 \in \mathrm{RingObj}(U_1)$.

(23) For every element x of $\mathrm{RingObj}(U_1)$ holds x is a strict ring.

Let us consider U_1. Then $\mathrm{RingObj}(U_1)$ is a non-empty set of rings.

Let us consider V. The functor $\mathrm{Morphs}\, V$ yielding a non-empty set of morphisms of rings is defined as follows:

(Def.18) $x \in \mathrm{Morphs}\, V$ if and only if there exist elements G, H of V such that $G \leq H$ and x is a morphism from G to H.

Let us consider V, and let F be an element of $\mathrm{Morphs}\,V$. Then $\mathrm{dom}\,F$ is an element of V. Then $\mathrm{cod}\,F$ is an element of V.

Let us consider V, and let G be an element of V. The functor I_G yields a strict element of $\mathrm{Morphs}\,V$ and is defined by:

(Def.19) $\mathrm{I}_G = \mathrm{I}_G$.

We now define three new functors. Let us consider V. The functor $\mathrm{dom}\,V$ yields a function from $\mathrm{Morphs}\,V$ into V and is defined as follows:

(Def.20) for every element f of $\mathrm{Morphs}\,V$ holds $(\mathrm{dom}\,V)(f) = \mathrm{dom}\,f$.

The functor $\mathrm{cod}\,V$ yielding a function from $\mathrm{Morphs}\,V$ into V is defined as follows:

(Def.21) for every element f of $\mathrm{Morphs}\,V$ holds $(\mathrm{cod}\,V)(f) = \mathrm{cod}\,f$.

The functor I_V yields a function from V into $\mathrm{Morphs}\,V$ and is defined by:

(Def.22) for every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

We now state two propositions:

(24) For all elements g, f of $\mathrm{Morphs}\,V$ such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ there exist elements G_1, G_2, G_3 of V such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(25) For all elements g, f of $\mathrm{Morphs}\,V$ such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ holds $g \cdot f \in \mathrm{Morphs}\,V$.

Let us consider V. The functor $\mathrm{comp}\,V$ yielding a partial function from $[\!:\mathrm{Morphs}\,V, \mathrm{Morphs}\,V:\!]$ to $\mathrm{Morphs}\,V$ is defined as follows:

(Def.23) for all elements g, f of $\mathrm{Morphs}\,V$ holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ if and only if $\mathrm{dom}\,g = \mathrm{cod}\,f$ and for all elements g, f of $\mathrm{Morphs}\,V$ such that $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ holds $(\mathrm{comp}\,V)(\langle g, f\rangle) = g \cdot f$.

Let us consider U_1. The functor $\mathrm{RingCat}(U_1)$ yielding a strict category structure is defined by:

(Def.24) $\mathrm{RingCat}(U_1) = \langle \mathrm{RingObj}(U_1), \mathrm{Morphs}\,\mathrm{RingObj}(U_1), \mathrm{dom}\,\mathrm{RingObj}(U_1), \mathrm{cod}\,\mathrm{RingObj}(U_1), \mathrm{comp}\,\mathrm{RingObj}(U_1), \mathrm{I}_{\mathrm{RingObj}(U_1)}\rangle$.

The following propositions are true:

(26) For all morphisms f, g of $\mathrm{RingCat}(U_1)$ holds $\langle g, f\rangle \in \mathrm{dom}\,(\text{the composition of } \mathrm{RingCat}(U_1))$ if and only if $\mathrm{dom}\,g = \mathrm{cod}\,f$.

(27) For every morphism f of $\mathrm{RingCat}(U_1)$ and for every element f' of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$
and for every object b of $\mathrm{RingCat}(U_1)$ and for every element b' of $\mathrm{RingObj}(U_1)$ holds f is a strict element of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$ and f' is a morphism of $\mathrm{RingCat}(U_1)$ and b is a strict element of $\mathrm{RingObj}(U_1)$ and b' is an object of $\mathrm{RingCat}(U_1)$.

(28) For every object b of $\mathrm{RingCat}(U_1)$ and for every element b' of $\mathrm{RingObj}(U_1)$ such that $b = b'$ holds
$\mathrm{id}_b = \mathrm{I}_{b'}$.

(29) For every morphism f of $\mathrm{RingCat}(U_1)$ and for every element f' of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$ such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(30) Let f, g be morphisms of $\mathrm{RingCat}(U_1)$. Let f', g' be elements of $\mathrm{Morphs}\,\mathrm{RingObj}(U_1)$. Suppose $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ if and only if $\langle g', f' \rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{RingObj}(U_1)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ if and only if $\mathrm{dom}\, f' = \mathrm{dom}\, g'$,

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ if and only if $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1. Then $\mathrm{RingCat}(U_1)$ is a strict category.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.

[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[8] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[9] Michał Muzalewski. Rings and modules - part II. *Formalized Mathematics*, 2(**4**):579–585, 1991.

[10] Michał Muzalewski and Wojciech Skaba. Groups, rings, left- and right-modules. *Formalized Mathematics*, 2(**2**):275–278, 1991.

[11] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.

[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[13] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received December 5, 1991