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On Powers of Cardinals

Grzegorz Bancerek
IM PAN, Warsaw
Warsaw University, Białystok

Summary. In the first section the results of [23, axiom (30)][1], i.e. the correspondence between natural and ordinal (cardinal) numbers are shown. The next section is concerned with the concepts of infinity and cofinality (see [3]), and introduces alephs as infinite cardinal numbers. The arithmetics of alephs, i.e. some facts about addition and multiplication, is present in the third section. The concepts of regular and irregular alephs are introduced in the fourth section, and the fact that $\aleph_0$ and every non-limit cardinal number are regular is proved there. Finally, for every alephs α and β

$$\alpha^\beta = \begin{cases} 2^\beta, & \text{if } \alpha \leq \beta, \\ \sum_{\gamma<\alpha} \gamma^\beta, & \text{if } \beta < \mathrm{cf}\alpha \text{ and } \alpha \text{ is limit cardinal,} \\ \left(\sum_{\gamma<\alpha} \gamma^\beta\right)^{\mathrm{cf}\alpha}, & \text{if } \mathrm{cf}\alpha \leq \beta \leq \alpha. \end{cases}$$

Some proofs are based on [20].

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The papers [24], [6], [16], [14], [21], [19], [26], [10], [17], [12], [15], [13], [25], [22], [11], [2], [18], [5], [9], [1], [8], [7], [4], and [3] provide the notation and terminology for this paper.

1. Results of [23, axiom (30)]

One can readily check that every set which is cardinal is also ordinal-like.

For simplicity we adopt the following convention: n denotes a natural number, A, B denote ordinal numbers, X denotes a set, and x, y are arbitrary. We now state several propositions:

[1]Axiom (30) – $n = \{k \in \mathbb{N} : k < n\}$ for every natural number n.

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(1) $0 = \emptyset$ and $1 = \{0\}$ and $2 = \{0, 1\}$.

(2) $\operatorname{succ} n = n + 1$.

(3) For every n holds $\operatorname{ord}(n) = n$ and $\overline{\overline{n}} = n$.

(4) $\mathbf{0} = 0$ and $\mathbf{1} = 1$.

(5) $\overline{\mathbf{0}} = 0$ and $\overline{\mathbf{1}} = 1$ and $\overline{\mathbf{2}} = 2$.

(6) If X is finite, then $\operatorname{card} X = \overline{\overline{X}}$.

(7) $\mathbb{N} = \omega$ and $\mathbb{N} = \aleph_{\mathbf{0}}$.

(8) $\operatorname{Seg} n = (n + 1) \setminus \{0\}$.

2. Infinity, alephs and cofinality

We adopt the following rules: f is a function, K, M, N are cardinal numbers, and p_1, p_2 are sequences of ordinal numbers. The following propositions are true:

(9) $\overline{\overline{X}}^{+} = X^{+}$.

(10) $y \in \bigcup f$ if and only if there exists x such that $x \in \operatorname{dom} f$ and $y \in f(x)$.

(11) $\aleph_A$ is not finite.

(12) If M is not finite, then there exists A such that $M = \aleph_A$.

(13) There exists n such that $M = \overline{\overline{n}}$ or there exists A such that $M = \aleph_A$.

Let us consider p_1. Then $\bigcup p_1$ is an ordinal number.

Next we state a number of propositions:

(14) If $X \subseteq A$, then there exists p_1 such that $p_1 =$ the canonical isomorphism between $\subseteq_{\overline{\subseteq_X}}$ and $\subseteq_X$ and p_1 is increasing and $\operatorname{dom} p_1 = \overline{\subseteq_X}$ and $\operatorname{rng} p_1 = X$.

(15) If $X \subseteq A$, then $\sup X$ is cofinal with $\overline{\subseteq_X}$.

(16) If $X \subseteq A$, then $\overline{\overline{X}} = \overline{\overline{\overline{\subseteq_X}}}$.

(17) There exists B such that $B \subseteq \overline{\overline{A}}$ and A is cofinal with B.

(18) There exists M such that $M \leq \overline{\overline{A}}$ and A is cofinal with M and for every B such that A is cofinal with B holds $M \subseteq B$.

(19) If $\operatorname{rng} p_1 = \operatorname{rng} p_2$ and p_1 is increasing and p_2 is increasing, then $p_1 = p_2$.

(20) If p_1 is increasing, then p_1 is one-to-one.

(21) $(p_1 \frown p_2) \restriction \operatorname{dom} p_1 = p_1$.

(22) If $X \neq \emptyset$, then $\overline{\overline{\{Y : \overline{\overline{Y}} < M\}}} \leq M \cdot \overline{\overline{X}}^{M}$, where Y ranges over elements of 2^X.

(23) $M < \overline{\mathbf{2}}^{M}$.

We now define four new constructions. A set is infinite if:

(Def.1) it is not finite.

Let us observe that there exists a set which is infinite. One can readily check that there exists a cardinal number which is infinite. One can readily check that every set which is infinite is also non-empty.

An aleph is an infinite cardinal number.

Let us consider M. The functor $\mathrm{cf}\,M$ yielding a cardinal number is defined by:

(Def.2) M is cofinal with $\mathrm{cf}\,M$ and for every N such that M is cofinal with N holds $\mathrm{cf}\,M \leq N$.

Let us consider N. The functor $(\alpha \mapsto \alpha^N)_{\alpha \in M}$ yielding a function yielding cardinal numbers is defined as follows:

(Def.3) for every x holds $x \in \mathrm{dom}((\alpha \mapsto \alpha^N)_{\alpha \in M})$ if and only if $x \in M$ and x is a cardinal number and for every K such that $K \in M$ holds $(\alpha \mapsto \alpha^N)_{\alpha \in M}(K) = K^N$.

Let us consider A. Then $\aleph_A$ is an aleph.

3. Arithmetics of alephs

In the sequel a, b will be alephs. The following propositions are true:

(24) There exists A such that $a = \aleph_A$.

(25) $a \neq \overline{\mathbf{0}}$ and $a \neq \overline{\mathbf{1}}$ and $a \neq \overline{\mathbf{2}}$ and $a \neq \overline{\overline{n}}$ and $\overline{\overline{n}} < a$ and $\aleph_{\mathbf{0}} \leq a$.

(26) If $a \leq M$ or $a < M$, then M is an aleph.

(27) If $a \leq M$ or $a < M$, then $a + M = M$ and $M + a = M$ and $a \cdot M = M$ and $M \cdot a = M$.

(28) $a + a = a$ and $a \cdot a = a$.

(29) If $M \leq a$ or $M < a$, then $a + M = a$ and $M + a = a$.

(30) If $\overline{\mathbf{0}} < M$ but $M \leq a$ or $M < a$, then $a \cdot M = a$ and $M \cdot a = a$.

(31) $M \leq M^a$.

(32) $\bigcup a = a$.

Let us consider a, M. Then $a + M$ is an aleph. Let us consider M, a. Then $M + a$ is an aleph. Let us consider a, b. Then $a + b$ is an aleph. Then $a \cdot b$ is an aleph. Then a^b is an aleph.

4. Regular alephs

We now define two new attributes. An aleph is regular if:

(Def.4) $\mathrm{cf}\,\mathrm{it} = \mathrm{it}$.

An aleph is irregular if:

(Def.5) $\mathrm{cf}\,\mathrm{it} < \mathrm{it}$.

Let us consider a. Then a^+ is an aleph. We see that the element of a is an ordinal number.

One can prove the following propositions:

(33) $\mathrm{cf}\, M \leq M$.

(34) $\mathrm{cf}(\aleph_{\mathbf{0}}) = \aleph_{\mathbf{0}}$.

(35) $\mathrm{cf}(a^+) = a^+$.

(36) $\aleph_{\mathbf{0}} \leq \mathrm{cf}\, a$.

(37) $\mathrm{cf}\, \overline{\mathbf{0}} = \overline{\mathbf{0}}$ and $\mathrm{cf}\, \overline{\overline{n+1}} = \overline{\mathbf{1}}$.

(38) If $X \subseteq M$ and $\overline{\overline{X}} < \mathrm{cf}\, M$, then $\sup X \in M$ and $\bigcup X \in M$.

(39) If $\mathrm{dom}\, p_1 = M$ and $\mathrm{rng}\, p_1 \subseteq N$ and $M < \mathrm{cf}\, N$, then $\sup p_1 \in N$ and $\bigcup p_1 \in N$.

Let us consider a. Then $\mathrm{cf}\, a$ is an aleph.

One can prove the following propositions:

(40) If $\mathrm{cf}\, a < a$, then a is a limit cardinal number.

(41) If $\mathrm{cf}\, a < a$, then there exists a sequence x_1 of ordinal numbers such that $\mathrm{dom}\, x_1 = \mathrm{cf}\, a$ and $\mathrm{rng}\, x_1 \subseteq a$ and x_1 is increasing and $a = \sup x_1$ and x_1 is a function yielding cardinal numbers and $\overline{\mathbf{0}} \notin \mathrm{rng}\, x_1$.

(42) $\aleph_{\mathbf{0}}$ is regular and a^+ is regular.

5. Infinite powers

In the sequel a, b will denote alephs. The following propositions are true:

(43) If $a \leq b$, then $a^b = \overline{\mathbf{2}}^b$.

(44) $(a^+)^b = a^b \cdot (a^+)$.

(45) $\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}) \leq a^b$.

(46) If a is a limit cardinal number and $b < \mathrm{cf}\, a$, then $a^b = \sum((\alpha \mapsto \alpha^b)_{\alpha \in a})$.

(47) If $\mathrm{cf}\, a \leq b$ and $b < a$, then $a^b = (\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}))^{\mathrm{cf}\, a}$.

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