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Oriented Metric-Affine Plane - Part II

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Summary. Axiomatic description of properties of the oriented orthogonality relation. Next we construct (with the help of the oriented orthogonality relation) vector space and give the definitions of left-, right-, and semi-transitives.

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The articles [1], [6], [7], [5], [3], [2], [4], and [8] provide the notation and terminology for this paper. In the sequel V will be a real linear space, A_1 will be an affine structure, and x, y will be vectors of V. One can prove the following propositions:

(1) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of CESpace(V, x, y) holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, w$, then $u, v \top^{>} u_1, w$ but if $u, u_1 \top^{>} v, v_1$, then $v, v_1 \top^{>} u, u_1$ or $v, v_1 \top^{>} u_1, u$,

(ii) for every elements u, v, w of the carrier of CESpace(V, x, y) there exists an element u_1 of the carrier of CESpace(V, x, y) such that $w \neq u_1$ and $w, u_1 \top^{>} u, v$,

(iii) for every elements u, v, w of the carrier of CESpace(V, x, y) there exists an element u_1 of the carrier of CESpace(V, x, y) such that $w \neq u_1$ and $u, v \top^{>} w, u_1$.

(2) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of CMSpace(V, x, y) holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$

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and $u, v \top^{>} v_1, w$, then $u, v \top^{>} u_1, w$ but if $u, u_1 \top^{>} v, v_1$, then $v, v_1 \top^{>} u, u_1$ or $v, v_1 \top^{>} u_1, u$,

(ii) for every elements u, v, w of the carrier of CMSpace(V, x, y) there exists an element u_1 of the carrier of CMSpace(V, x, y) such that $w \neq u_1$ and $w, u_1 \top^{>} u, v$,

(iii) for every elements u, v, w of the carrier of CMSpace(V, x, y) there exists an element u_1 of the carrier of CMSpace(V, x, y) such that $w \neq u_1$ and $u, v \top^{>} w, u_1$.

We now define two new constructions. An affine structure is oriented orthogonality if it satisfies the conditions (Def.1).

(Def.1) (i) For all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of it holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, w$, then $u, v \top^{>} u_1, w$ but if $u, u_1 \top^{>} v, v_1$, then $v, v_1 \top^{>} u, u_1$ or $v, v_1 \top^{>} u_1, u$,

(ii) for every elements u, v, w of the carrier of it there exists an element u_1 of the carrier of it such that $w \neq u_1$ and $w, u_1 \top^{>} u, v$,

(iii) for every elements u, v, w of the carrier of it there exists an element u_1 of the carrier of it such that $w \neq u_1$ and $u, v \top^{>} w, u_1$.

An oriented orthogonality space is an oriented orthogonality affine structure.

Next we state three propositions:

(3) The following conditions are equivalent:

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of A_1 holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, w$, then $u, v \top^{>} u_1, w$ but if $u, u_1 \top^{>} v, v_1$, then $v, v_1 \top^{>} u, u_1$ or $v, v_1 \top^{>} u_1, u$ and for every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $w \neq u_1$ and $w, u_1 \top^{>} u, v$ and for every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $w \neq u_1$ and $u, v \top^{>} w, u_1$,

(ii) A_1 is an oriented orthogonality space.

(4) If x, y span the space, then CMSpace(V, x, y) is an oriented orthogonality space.

(5) If x, y span the space, then CESpace(V, x, y) is an oriented orthogonality space.

We follow a convention: A_1 will denote an oriented orthogonality space and u, u_1, u_2, v, v_1, v_2, w, w_1 will denote elements of the carrier of A_1. We now state three propositions:

(6) For every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $u_1, w \top^{>} u, v$ and $u_1 \neq w$.

(7) For all elements u, v, w of the carrier of A_1 holds $u, v \top^{>} w, w$.

(8) For every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $u \neq u_1$ but $v, w \top^{>} u, u_1$ or $v, w \top^{>} u_1, u$.

We now define several new constructions. Let A_1 be an oriented orthogonality space, and let a, b, c, d be elements of the carrier of A_1. The predicate $a, b \perp c, d$ is defined by:

(Def.2) $a, b \top^{>} c, d$ or $a, b \top^{>} d, c$.

Let a, b, c, d be elements of the carrier of A_1. The predicate $a, b \upharpoonleft\!\!\upharpoonright c, d$ is defined as follows:

(Def.3) there exist elements e, f of the carrier of A_1 such that $e \neq f$ and $e, f \top^{>} a, b$ and $e, f \top^{>} c, d$.

An oriented orthogonality space is semi transitive if:

(Def.4) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $w, w_1 \top^{>} v, v_1$ and $w, w_1 \top^{>} u_2, v_2$ holds $w = w_1$ or $v = v_1$ or $u, u_1 \top^{>} u_2, v_2$.

An oriented orthogonality space is right transitive if:

(Def.5) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \top^{>} w, w_1$ and $u_2, v_2 \top^{>} w, w_1$ holds $w = w_1$ or $v = v_1$ or $u, u_1 \top^{>} u_2, v_2$.

An oriented orthogonality space is left transitive if:

(Def.6) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \top^{>} w, w_1$ and $u, u_1 \top^{>} u_2, v_2$ holds $u = u_1$ or $v = v_1$ or $u_2, v_2 \top^{>} w, w_1$.

An oriented orthogonality space is Euclidean like if:

(Def.7) for all elements u, u_1, v, v_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ holds $v, v_1 \top^{>} u_1, u$.

An oriented orthogonality space is Minkowskian like if:

(Def.8) for all elements u, u_1, v, v_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ holds $v, v_1 \top^{>} u, u_1$.

One can prove the following propositions:

(9) $u, u_1 \upharpoonleft\!\!\upharpoonright w, w$ and $w, w \upharpoonleft\!\!\upharpoonright u, u_1$.

(10) If $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$, then $v, v_1 \upharpoonleft\!\!\upharpoonright u, u_1$.

(11) If $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$, then $u_1, u \upharpoonleft\!\!\upharpoonright v_1, v$.

(12) A_1 is left transitive if and only if for all v, v_1, w, w_1, u_2, v_2 such that $v, v_1 \upharpoonleft\!\!\upharpoonright u_2, v_2$ and $v, v_1 \top^{>} w, w_1$ and $v \neq v_1$ holds $u_2, v_2 \top^{>} w, w_1$.

(13) A_1 is semi transitive if and only if for all u, u_1, u_2, v, v_1, v_2 such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \upharpoonleft\!\!\upharpoonright u_2, v_2$ and $v \neq v_1$ holds $u, u_1 \top^{>} u_2, v_2$.

(14) If A_1 is semi transitive, then for all u, u_1, v, v_1, w, w_1 such that $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$ and $v, v_1 \upharpoonleft\!\!\upharpoonright w, w_1$ and $v \neq v_1$ holds $u, u_1 \upharpoonleft\!\!\upharpoonright w, w_1$.

(15) If x, y span the space and $A_1 = \mathrm{CESpace}(V, x, y)$, then A_1 is Euclidean like, left transitive, right transitive and semi transitive.

One can readily verify that there exists an oriented orthogonality space which is Euclidean like, left transitive, right transitive and semi transitive.

We now state the proposition

(16) If x, y span the space and $A_1 = \mathrm{CMSpace}(V, x, y)$, then A_1 is Minkowskian like, left transitive, right transitive and semi transitive.

Let us note that there exists an oriented orthogonality space which is Minkowskian like, left transitive, right transitive and semi transitive.

Next we state four propositions:

(17) If A_1 is left transitive, then A_1 is right transitive.

(18) If A_1 is left transitive, then A_1 is semi transitive.

(19) If A_1 is semi transitive, then A_1 is right transitive if and only if for all u, u_1, v, v_1, u_2, v_2 such that $u, u_1 \top^{>} u_2, v_2$ and $v, v_1 \top^{>} u_2, v_2$ and $u_2 \neq v_2$ holds $u, u_1 \upuparrows v, v_1$.

(20) If A_1 is right transitive but A_1 is Euclidean like or A_1 is Minkowskian like, then A_1 is left transitive.

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