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Introduction to Go-Board - Part I

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Summary. In the article we introduce Go-board as some kinds of matrix which elements belong to topological space $\mathcal{E}^2_{\rm T}$. We define the functor of delaying column in Go-board and relation between Go-board and finite sequence of point from $\mathcal{E}^2_{\rm T}$. Basic facts about those notations are proved. The concept of the article is based on [16].

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The notation and terminology used here have been introduced in the following papers: [17], [11], [2], [6], [3], [9], [7], [14], [15], [1], [18], [5], [12], [4], [8], [10], and [13].

1. Real Numbers Preliminaries

For simplicity we follow the rules: p denotes a point of $\mathcal{E}^2_{\rm T}$, f, f_1, f_2, g denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, v denotes a finite sequence of elements of $\mathbb{R}$, r, s denote real numbers, n, m, i, j, k denote natural numbers, and x is arbitrary. One can prove the following three propositions:

(1) $|r-s|=1$ if and only if $r>s$ and $r=s+1$ or $r<s$ and $s=r+1$.

(2) $|i-j|+|n-m|=1$ if and only if $|i-j|=1$ and $n=m$ or $|n-m|=1$ and $i=j$.

(3) $n>1$ if and only if there exists m such that $n=m+1$ and $m>0$.

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2. Finite Sequences Preliminaries

The scheme *FinSeqDChoice* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a finite sequence f of elements of $\mathcal{A}$ such that $\operatorname{len} f = \mathcal{B}$ and for every n such that $n \in \operatorname{Seg} \mathcal{B}$ holds $\mathcal{P}[n, f(n)]$

provided the parameters have the following property:

- for every n such that $n \in \operatorname{Seg} \mathcal{B}$ there exists an element d of $\mathcal{A}$ such that $\mathcal{P}[n, d]$.

One can prove the following propositions:

(4) If $n = m + 1$ and $i \in \operatorname{Seg} n$, then $\operatorname{len} \operatorname{Sgm}(\operatorname{Seg} n \setminus \{i\}) = m$.

(5) Suppose $n = m + 1$ and $k \in \operatorname{Seg} n$ and $i \in \operatorname{Seg} m$. Then if $1 \leq i$ and $i < k$, then $(\operatorname{Sgm}(\operatorname{Seg} n \setminus \{k\}))(i) = i$ but if $k \leq i$ and $i \leq m$, then $(\operatorname{Sgm}(\operatorname{Seg} n \setminus \{k\}))(i) = i + 1$.

(6) For every finite sequence f and for all n, m such that $\operatorname{len} f = m+1$ and $n \in \operatorname{Seg} \operatorname{len} f$ holds $\operatorname{len}(f_{\restriction n}) = m$.

(7) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $k \in \operatorname{Seg} m$ holds $f_{\restriction n}(k) = f(k)$ or $f_{\restriction n}(k) = f(k+1)$.

(8) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $1 \leq k$ and $k < n$ holds $f_{\restriction n}(k) = f(k)$.

(9) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $n \leq k$ and $k \leq m$ holds $f_{\restriction n}(k) = f(k + 1)$.

(10) If $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$, then $(f \restriction n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

We now define four new constructions. A finite sequence of elements of $\mathbb{R}$ is increasing if:

(Def.1) for all n, m such that $n \in \operatorname{dom} \mathrm{it}$ and $m \in \operatorname{dom} \mathrm{it}$ and $n < m$ and for all r, s such that $r = \mathrm{it}(n)$ and $s = \mathrm{it}(m)$ holds $r < s$.

A finite sequence is constant if:

(Def.2) for all n, m such that $n \in \operatorname{dom} \mathrm{it}$ and $m \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(n) = \mathrm{it}(m)$.

Let us observe that there exists a finite sequence of elements of $\mathbb{R}$ which is increasing. Note also that there exists a finite sequence of elements of $\mathbb{R}$ which is constant.

Let us consider f. The functor $\mathbf{X}$-coordinate(f) yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.3) $\operatorname{len} \mathbf{X}\text{-coordinate}(f) = \operatorname{len} f$
and for every n such that $n \in \operatorname{dom} \mathbf{X}\text{-coordinate}(f)$ and for every p such that $p = f(n)$ holds $(\mathbf{X}\text{-coordinate}(f))(n) = p_{\mathbf{1}}$.

The functor $\mathbf{Y}$-coordinate(f) yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def.4) $\operatorname{len} \mathbf{Y}\text{-coordinate}(f) = \operatorname{len} f$
and for every n such that $n \in \operatorname{dom} \mathbf{Y}\text{-coordinate}(f)$ and for every p such that $p = f(n)$ holds $(\mathbf{Y}\text{-coordinate}(f))(n) = p_{\mathbf{2}}$.

One can prove the following propositions:

(11) Suppose that
(i) $v \neq \varepsilon$,
(ii) $\operatorname{rng} v \subseteq \operatorname{Seg} n$,
(iii) $v(\operatorname{len} v) = n$,
(iv) for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r - s| = 1$ or $r = s$,
(v) $i \in \operatorname{Seg} n$,
(vi) $i + 1 \in \operatorname{Seg} n$,
(vii) $m \in \operatorname{dom} v$,
(viii) $v(m) = i$,
(ix) for every k such that $k \in \operatorname{dom} v$ and $v(k) = i$ holds $k \leq m$.
Then $m + 1 \in \operatorname{dom} v$ and $v(m+1) = i + 1$.

(12) Suppose that
(i) $v \neq \varepsilon$,
(ii) $\operatorname{rng} v \subseteq \operatorname{Seg} n$,
(iii) $v(1) = 1$,
(iv) $v(\operatorname{len} v) = n$,
(v) for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r - s| = 1$ or $r = s$.
Then
(vi) for every i such that $i \in \operatorname{Seg} n$ there exists k such that $k \in \operatorname{dom} v$ and $v(k) = i$,
(vii) for all m, k, i, r such that $m \in \operatorname{dom} v$ and $v(m) = i$ and for every j such that $j \in \operatorname{dom} v$ and $v(j) = i$ holds $j \leq m$ and $m < k$ and $k \in \operatorname{dom} v$ and $r = v(k)$ holds $i < r$.

(13) If $i \in \operatorname{dom} f$ and $2 \leq \operatorname{len} f$, then $f(i) \in \widetilde{\mathcal{L}}(f)$.

3. Matrix Preliminaries

Next we state two propositions:

(14) For every non-empty set D and for every matrix M over D and for all i, j such that $j \in \operatorname{Seg} \operatorname{len} M$ and $i \in \operatorname{Seg} \operatorname{width} M$ holds $M_{\square,i}(j) = \operatorname{Line}(M, j)(i)$.

(15) For every non-empty set D and for every matrix M over D and for every k such that $k \in \operatorname{Seg} \operatorname{len} M$ holds $M(k) = \operatorname{Line}(M, k)$.

We now define several new constructions. Let T be a topological space. A matrix over T is a matrix over the carrier of T.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is non-trivial if:

(Def.5) $0 < \operatorname{len} \mathrm{it}$ and $0 < \operatorname{width} \mathrm{it}$.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is line $\mathbf{X}$-constant if:

(Def.6) for every n such that $n \in \mathrm{Seg}\,\mathrm{len}\,\mathrm{it}$ holds **X**-coordinate(Line(it, n)) is constant.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is column **Y**-constant if:

(Def.7) for every n such that $n \in \mathrm{Seg}\,\mathrm{width}\,\mathrm{it}$ holds **Y**-coordinate($\mathrm{it}_{\square,n}$) is constant.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is line **Y**-increasing if:

(Def.8) for every n such that $n \in \mathrm{Seg}\,\mathrm{len}\,\mathrm{it}$ holds **Y**-coordinate(Line(it, n)) is increasing.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is column **X**-increasing if:

(Def.9) for every n such that $n \in \mathrm{Seg}\,\mathrm{width}\,\mathrm{it}$ holds **X**-coordinate($\mathrm{it}_{\square,n}$) is increasing.

One can readily verify that there exists a matrix over $\mathcal{E}^2_{\mathrm{T}}$ which is non-trivial, line **X**-constant, column **Y**-constant, line **Y**-increasing and column **X**-increasing.

We now state two propositions:

(16) For every column **X**-increasing line **X**-constant matrix M over $\mathcal{E}^2_{\mathrm{T}}$ and for all x, n, m such that $x \in \mathrm{rng}\,\mathrm{Line}(M, n)$ and $x \in \mathrm{rng}\,\mathrm{Line}(M, m)$ and $n \in \mathrm{Seg}\,\mathrm{len}\,M$ and $m \in \mathrm{Seg}\,\mathrm{len}\,M$ holds $n = m$.

(17) For every line **Y**-increasing column **Y**-constant matrix M over $\mathcal{E}^2_{\mathrm{T}}$ and for all x, n, m such that $x \in \mathrm{rng}(M_{\square,n})$ and $x \in \mathrm{rng}(M_{\square,m})$ and $n \in \mathrm{Seg}\,\mathrm{width}\,M$ and $m \in \mathrm{Seg}\,\mathrm{width}\,M$ holds $n = m$.

4. Basic Go-Board's Notation

A Go-board is a non-trivial line **X**-constant column **Y**-constant line **Y**-increasing column **X**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$.

In the sequel G denotes a Go-board. The following four propositions are true:

(18) If $x = G_{m,k}$ and $x = G_{i,j}$ and $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G, then $m = i$ and $k = j$.

(19) If $m \in \mathrm{dom}\,f$ and $f(1) \in \mathrm{rng}(G_{\square,1})$, then $(f \restriction m)(1) \in \mathrm{rng}(G_{\square,1})$.

(20) If $m \in \mathrm{dom}\,f$ and $f(m) \in \mathrm{rng}(G_{\square,\mathrm{width}\,G})$, then $(f \restriction m)(\mathrm{len}(f \restriction m)) \in \mathrm{rng}(G_{\square,\mathrm{width}\,G})$.

(21) If $\mathrm{rng}\,f \cap \mathrm{rng}(G_{\square,i}) = \emptyset$ and $f(n) = G_{m,k}$ and $n \in \mathrm{dom}\,f$ and $m \in \mathrm{Seg}\,\mathrm{len}\,G$, then $i \neq k$.

Let us consider G, i. Let us assume that $i \in \mathrm{Seg}\,\mathrm{width}\,G$ and $\mathrm{width}\,G > 1$. The deleting of i-column in G yielding a Go-board is defined by:

(Def.10) len(the deleting of i-column in G) $= \mathrm{len}\,G$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\,G$ holds (the deleting of i-column in G)$(k) = \mathrm{Line}(G, k)_{\restriction i}$.

One can prove the following propositions:

(22) If $i \in \mathrm{Seg}\,\mathrm{width}\,G$ and $\mathrm{width}\,G > 1$ and $k \in \mathrm{Seg}\,\mathrm{len}\,G$, then Line(the deleting of i-column in $G, k) = \mathrm{Line}(G, k)_{\restriction i}$.

(23) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$, then width(the deleting of i-column in G) $= m$.

(24) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$, then $\operatorname{width} G =$ width(the deleting of i-column in G) $+ 1$.

(25) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $m \in$ Seg width(the deleting of i-column in G), then (the deleting of i-column in $G)_{n,m} = \operatorname{Line}(G, n)_{\restriction i}(m)$.

(26) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $1 \leq k$ and $k < i$, then (the deleting of i-column in $G)_{\square,k} = G_{\square,k}$ and $k \in$ Seg width(the deleting of i-column in G) and $k \in \operatorname{Seg} \operatorname{width} G$.

(27) Suppose $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in$ Seg width(the deleting of i-column in G) and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(28) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $1 \leq k$ and $k < i$, then (the deleting of i-column in $G)_{n,k} = G_{n,k}$ and $k \in \operatorname{Seg} \operatorname{width} G$.

(29) Suppose $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in $G)_{n,k} = G_{n,k+1}$ and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(30) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$, then (the deleting of 1-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in$ Seg width(the deleting of 1-column in G) and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(31) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{Seg} \operatorname{len} G$, then (the deleting of 1-column in $G)_{n,k} = G_{n,k+1}$ and $1 \in \operatorname{Seg} \operatorname{width} G$.

(32) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$, then (the deleting of $\operatorname{width} G$-column in $G)_{\square,k} = G_{\square,k}$ and $k \in$ Seg width(the deleting of $\operatorname{width} G$-column in G).

(33) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{Seg} \operatorname{len} G$, then $k \in \operatorname{Seg} \operatorname{width} G$ and (the deleting of $\operatorname{width} G$-column in $G)_{n,k} = G_{n,k}$ and $\operatorname{width} G \in \operatorname{Seg} \operatorname{width} G$.

(34) Suppose $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,i}) = \emptyset$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, m)$ and $n \in \operatorname{dom} f$ and $i \in \operatorname{Seg} \operatorname{width} G$ and $m \in \operatorname{Seg} \operatorname{len} G$ and $\operatorname{width} G > 1$. Then $f(n) \in$ rng Line(the deleting of i-column in G, m).

Let us consider f, G. We say that f is a sequence which elements belong to G if and only if the conditions (Def.11) is satisfied.

(Def.11) (i) For every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j \rangle \in$ the indices of G and $f(n) = G_{i,j}$,

(ii) for every n such that $n \in \operatorname{dom} f$ and $n + 1 \in \operatorname{dom} f$ and for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G and $f(n) = G_{m,k}$ and $f(n + 1) = G_{i,j}$ holds $|m - i| + |k - j| = 1$.

One can prove the following propositions:

(35) If f is a sequence which elements belong to G and $m \in \operatorname{dom} f$, then $1 \leq \operatorname{len}(f \restriction m)$ and $f \restriction m$ is a sequence which elements belong to G.

(36) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ there exist i, j such that $\langle i, j \rangle \in$ the indices of G and $f_1(n) = G_{i,j}$,

(ii) for every n such that $n \in \operatorname{dom} f_2$ there exist i, j such that $\langle i, j \rangle \in$ the indices of G and $f_2(n) = G_{i,j}$.

Then for every n such that $n \in \operatorname{dom}(f_1 \frown f_2)$ there exist i, j such that $\langle i, j \rangle \in$ the indices of G and $(f_1 \frown f_2)(n) = G_{i,j}$.

(37) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ and for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G and $f_1(n) = G_{m,k}$ and $f_1(n+1) = G_{i,j}$ holds $|m - i| + |k - j| = 1$,

(ii) for every n such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ and for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G and $f_2(n) = G_{m,k}$ and $f_2(n+1) = G_{i,j}$ holds $|m - i| + |k - j| = 1$,

(iii) for all m, k, i, j such that $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G and $f_1(\operatorname{len} f_1) = G_{m,k}$ and $f_2(1) = G_{i,j}$ and $\operatorname{len} f_1 \in \operatorname{dom} f_1$ and $1 \in \operatorname{dom} f_2$ holds $|m - i| + |k - j| = 1$.

Given n. Suppose $n \in \operatorname{dom}(f_1 \frown f_2)$ and $n+1 \in \operatorname{dom}(f_1 \frown f_2)$. Given m, k, i, j. Then if $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G and $(f_1 \frown f_2)(n) = G_{m,k}$ and $(f_1 \frown f_2)(n+1) = G_{i,j}$, then $|m-i|+|k-j| = 1$.

(38) If f is a sequence which elements belong to G and $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,i}) = \emptyset$ and $\operatorname{width} G > 1$, then f is a sequence which elements belong to the deleting of i-column in G.

(39) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{Seg} \operatorname{len} G$ and $f(i) \in \operatorname{rng} \operatorname{Line}(G, n)$.

(40) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i + 1 \in \operatorname{dom} f$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $f(i) \in \operatorname{rng} \operatorname{Line}(G, n)$. Then $f(i+1) \in \operatorname{rng} \operatorname{Line}(G, n)$ or for every k such that $f(i+1) \in \operatorname{rng} \operatorname{Line}(G, k)$ and $k \in \operatorname{Seg} \operatorname{len} G$ holds $|n - k| = 1$.

(41) Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) $f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,

(iii) f is a sequence which elements belong to G,

(iv) $i \in \operatorname{Seg} \operatorname{len} G$,

(v) $i + 1 \in \operatorname{Seg} \operatorname{len} G$,

(vi) $m \in \operatorname{dom} f$,

(vii) $f(m) \in \operatorname{rng} \operatorname{Line}(G, i)$,

(viii) for every k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $k \leq m$.

Then $m + 1 \in \operatorname{dom} f$ and $f(m + 1) \in \operatorname{rng} \operatorname{Line}(G, i + 1)$.

(42) Suppose $1 \leq \operatorname{len} f$ and $f(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$ and
$f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$
and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ there exists k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$,

(ii) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $2 \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(f) \cap \operatorname{rng} \operatorname{Line}(G, i) \neq \emptyset$,

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{len} G$ and $k \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$ and for every n such that $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $n \leq k$ and $k < m$ and $f(m) \in \operatorname{rng} \operatorname{Line}(G, j)$ holds $i < j$.

(43) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,n})$.

(44) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $n \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,n})$. Then $f(i+1) \in \operatorname{rng}(G_{\square,n})$ or for every k such that $f(i+1) \in \operatorname{rng}(G_{\square,k})$ and $k \in \operatorname{Seg} \operatorname{width} G$ holds $|n-k| = 1$.

(45) Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$,

(iii) f is a sequence which elements belong to G,

(iv) $i \in \operatorname{Seg} \operatorname{width} G$,

(v) $i+1 \in \operatorname{Seg} \operatorname{width} G$,

(vi) $m \in \operatorname{dom} f$,

(vii) $f(m) \in \operatorname{rng}(G_{\square,i})$,

(viii) for every k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$ holds $k \leq m$.

Then $m+1 \in \operatorname{dom} f$ and $f(m+1) \in \operatorname{rng}(G_{\square,i+1})$.

(46) Suppose $1 \leq \operatorname{len} f$ and $f(1) \in \operatorname{rng}(G_{\square,1})$ and $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \operatorname{width} G$ there exists k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$,

(ii) for every i such that $1 \leq i$ and $i \leq \operatorname{width} G$ and $2 \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(f) \cap \operatorname{rng}(G_{\square,i}) \neq \emptyset$,

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \operatorname{width} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$ and $k \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$ and for every n such that $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng}(G_{\square,i})$ holds $n \leq k$ and $k < m$ and $f(m) \in \operatorname{rng}(G_{\square,j})$ holds $i < j$.

(47) Suppose that

(i) $n \in \operatorname{dom} f$,

(ii) $f(n) \in \operatorname{rng}(G_{\square,k})$,

(iii) $k \in \operatorname{Seg} \operatorname{width} G$,

(iv) $f(1) \in \operatorname{rng}(G_{\square,1})$,

(v) f is a sequence which elements belong to G,

(vi) for every i such that $i \in \operatorname{dom} f$ and $f(i) \in \operatorname{rng}(G_{\square,k})$ holds $n \leq i$.

Then for every i such that $i \in \operatorname{dom} f$ and $i \leq n$ and for every m such that $m \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,m})$ holds $m \leq k$.

(48) Suppose f is a sequence which elements belong to G and $f(1) \in \operatorname{rng}(G_{\square,1})$ and $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and $\operatorname{width} G > 1$ and $1 \leq \operatorname{len} f$. Then there exists g such that $g(1) \in \operatorname{rng}((\text{the deleting of } \operatorname{width} G\text{-column in } G)_{\square,1})$ and $g(\operatorname{len} g) \in \operatorname{rng}((\text{the deleting of } \operatorname{width} G\text{-column in } G)_{\square,\operatorname{width}(\text{the deleting of } \operatorname{width} G-\text{column in } G)})$ and $1 \leq \operatorname{len} g$ and g is a sequence which elements belong to the deleting of $\operatorname{width} G$-column in G and $\operatorname{rng} g \subseteq \operatorname{rng} f$.

(49) Suppose f is a sequence which elements belong to G and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,1}) \neq \emptyset$ and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,\operatorname{width} G}) \neq \emptyset$. Then there exists g such that $\operatorname{rng} g \subseteq \operatorname{rng} f$ and $g(1) \in \operatorname{rng}(G_{\square,1})$ and $g(\operatorname{len} g) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and $1 \leq \operatorname{len} g$ and g is a sequence which elements belong to G.

(50) Suppose $k \in \operatorname{Seg} \operatorname{len} G$ and f is a sequence which elements belong to G and $f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$ and $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, k)$. Then

(i) for every i such that $k \leq i$ and $i \leq \operatorname{len} G$ there exists j such that $j \in \operatorname{dom} f$ and $n \leq j$ and $f(j) \in \operatorname{rng} \operatorname{Line}(G, i)$,

(ii) for every i such that $k < i$ and $i \leq \operatorname{len} G$ there exists j such that $j \in \operatorname{dom} f$ and $n < j$ and $f(j) \in \operatorname{rng} \operatorname{Line}(G, i)$.

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