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Introduction to Go-Board - Part II

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Summary. In article we define Go-board determined by finite sequence of points from topological space $\mathcal{E}^2_{\rm T}$. A few facts about this notation are proved.

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The papers [17], [10], [2], [6], [3], [8], [15], [16], [1], [18], [13], [5], [12], [11], [4], [7], [9], and [14] provide the notation and terminology for this paper.

1. Real Numbers Preliminaries

For simplicity we follow the rules: p, q denote points of $\mathcal{E}^2_{\rm T}$, f, f_1, f_2, g denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, R denotes a subset of $\mathbb{R}$, r, s denote real numbers, v, v_1, v_2 denote finite sequences of elements of $\mathbb{R}$, n, m, i, j, k denote natural numbers, and G denotes a Go-board. We now state the proposition

(1) If R is finite and $R \neq \emptyset$, then R is upper bounded and $\sup R \in R$ and R is lower bounded and $\inf R \in R$.

2. Properties of Finite Sequences of Points from $\mathcal{E}^2_{\rm T}$

One can prove the following propositions:

(2) For every finite sequence f holds f is one-to-one if and only if for all n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $n \neq m$ holds $f(n) \neq f(m)$.

(3) For every n holds $1 \leq n$ and $n \leq \operatorname{len} f - 1$ if and only if $n \in \operatorname{dom} f$ and $n + 1 \in \operatorname{dom} f$.

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(4) For every n holds $1 \leq n$ and $n \leq \operatorname{len} f - 2$ if and only if $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $n+2 \in \operatorname{dom} f$.

(5) The following conditions are equivalent:

(i) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$,

(ii) for all n, m such that $n - m > 1$ or $m - n > 1$ but $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $m+1 \in \operatorname{dom} f$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$.

(6) Suppose that

(i) for every n such that $1 \leq n$ and $n \leq \operatorname{len} f - 2$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, n+1, n+2) = \{f(n+1)\}$,

(ii) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$,

(iii) f is one-to-one,

(iv) $f(\operatorname{len} f) \in \mathcal{L}(f, i, i+1)$,

(v) $i \in \operatorname{dom} f$,

(vi) $i + 1 \in \operatorname{dom} f$.

Then $i + 1 = \operatorname{len} f$.

(7) If $k \neq 0$ and $\operatorname{len} f = k + 1$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(f \restriction k) \cup \mathcal{L}(f, k, k+1)$.

(8) Suppose that

(i) $1 < k$,

(ii) $\operatorname{len} f = k + 1$,

(iii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} f - 2$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, n+1, n+2) = \{f(n+1)\}$,

(iv) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$.

Then $\widetilde{\mathcal{L}}(f \restriction k) \cap \mathcal{L}(f, k, k+1) = \{f(k)\}$.

(9) If $\operatorname{len} f_1 < n$ and $n \leq \operatorname{len}(f_1 \frown f_2) - 1$ and $m = n - \operatorname{len} f_1$, then $\mathcal{L}(f_1 \frown f_2, n, n+1) = \mathcal{L}(f_2, m, m+1)$.

(10) $\widetilde{\mathcal{L}}(f) \subseteq \widetilde{\mathcal{L}}(f \frown g)$.

(11) Suppose for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$. Then for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f \restriction i, n, n+1) \cap \mathcal{L}(f \restriction i, m, m+1) = \emptyset$.

(12) Suppose that

(i) for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} f_1 - 1$ and $f_1(n) = p$ and $f_1(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(ii) for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} f_2 - 1$ and $f_2(n) = p$ and $f_2(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(iii) for all p, q such that $f_1(\operatorname{len} f_1) = p$ and $f_2(1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$.

Then for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len}(f_1 \frown f_2) - 1$ and $(f_1 \frown f_2)(n) = p$ and $(f_1 \frown f_2)(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$.

(13) If $f \neq \varepsilon$, then $\mathbf{X}$-coordinate$(f) \neq \varepsilon$.

(14) If $f \neq \varepsilon$, then **Y**-coordinate$(f) \neq \varepsilon$.

(15) Suppose for all n, p, q such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $f(n) = p$ and $f(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$. Given n. Suppose $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$. Then for all i, j, m, k such that $\langle i, j\rangle \in$ the indices of G and $\langle m, k\rangle \in$ the indices of G and $f(n) = G_{i,j}$ and $f(n+1) = G_{m,k}$ holds $i = m$ or $k = j$.

(16) Suppose that

(i) for every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{i,j}$,

(ii) for all n, p, q such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $f(n) = p$ and $f(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(iii) for every n such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ holds $f(n) \neq f(n+1)$.

Then there exists g such that g is a sequence which elements belong to G and $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(g)$ and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$ and $\operatorname{len} f \leq \operatorname{len} g$.

(17) If v is increasing, then for all n, m such that $n \in \operatorname{dom} v$ and $m \in \operatorname{dom} v$ and $n \leq m$ and for all r, s such that $r = v(n)$ and $s = v(m)$ holds $r \leq s$.

(18) If v is increasing, then for all n, m such that $n \in \operatorname{dom} v$ and $m \in \operatorname{dom} v$ and $n \neq m$ holds $v(n) \neq v(m)$.

(19) If v is increasing and $v_1 = v \restriction \operatorname{Seg} n$, then v_1 is increasing.

(20) For every v there exists v_1 such that $\operatorname{rng} v_1 = \operatorname{rng} v$ and $\operatorname{len} v_1 = \operatorname{card} \operatorname{rng} v$ and v_1 is increasing.

(21) For all v_1, v_2 such that $\operatorname{len} v_1 = \operatorname{len} v_2$ and $\operatorname{rng} v_1 = \operatorname{rng} v_2$ and v_1 is increasing and v_2 is increasing holds $v_1 = v_2$.

3. Go-Board Determined by Finite Sequence

We now define three new functors. Let v_1, v_2 be increasing finite sequences of elements of $\mathbb{R}$. Let us assume that $v_1 \neq \varepsilon$ and $v_2 \neq \varepsilon$. The Go-board of v_1, v_2 yields a Go-board and is defined by:

(Def.1) len the Go-board of v_1, $v_2 = \operatorname{len} v_1$ and width the Go-board of v_1, $v_2 = \operatorname{len} v_2$ and for all n, m such that $\langle n, m\rangle \in$ the indices of the Go-board of v_1, v_2 and for all r, s such that $v_1(n) = r$ and $v_2(m) = s$ holds (the Go-board of v_1, $v_2)_{n,m} = [r, s]$.

Let us consider v. The functor $\operatorname{Inc}(v)$ yielding an increasing finite sequence of elements of $\mathbb{R}$ is defined by:

(Def.2) $\operatorname{rng} \operatorname{Inc}(v) = \operatorname{rng} v$ and $\operatorname{len} \operatorname{Inc}(v) = \operatorname{card} \operatorname{rng} v$.

Let us consider f. Let us assume that $f \neq \varepsilon$. The Go-board of f yielding a Go-board is defined by:

(Def.3) the Go-board of $f =$ the Go-board of Inc(**X**-coordinate(f)), Inc(**Y**-coordinate(f)).

One can prove the following propositions:

(22) If $v \neq \varepsilon$, then $\mathrm{Inc}(v) \neq \varepsilon$.

(23) If $f \neq \varepsilon$, then len the Go-board of f = card rng **X**-coordinate(f) and width the Go-board of f = card rng **Y**-coordinate(f).

(24) If $f \neq \varepsilon$, then for every n such that $n \in \mathrm{dom}\, f$ there exist i, j such that $\langle i, j \rangle \in$ the indices of the Go-board of f and $f(n) =$ (the Go-board of $f)_{i,j}$.

(25) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{X}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{X}\text{-coordinate}(f))(m)$ holds $r \leq s$, then $f(n) \in \mathrm{rng}\,\mathrm{Line}(\text{the Go-board of } f, 1)$.

(26) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{X}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{X}\text{-coordinate}(f))(m)$ holds $s \leq r$, then $f(n) \in \mathrm{rng}\,\mathrm{Line}(\text{the Go-board of } f, \mathrm{len}\,\text{the Go-board of } f)$.

(27) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{Y}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{Y}\text{-coordinate}(f))(m)$ holds $r \leq s$, then $f(n) \in \mathrm{rng}((\text{the Go-board of } f)_{\Box,1})$.

(28) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{Y}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{Y}\text{-coordinate}(f))(m)$ holds $s \leq r$, then $f(n) \in \mathrm{rng}((\text{the Go-board of } f)_{\Box,\mathrm{width\,the\,Go-board\,of}\,f})$.

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