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Go-Board Theorem

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Summary. We prove the Go-board theorem which is a special case of Hex Theorem. The article is based on [15].

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The terminology and notation used in this paper are introduced in the following articles: [16], [7], [1], [4], [2], [13], [14], [17], [3], [8], [5], [6], [9], [12], [10], and [11]. For simplicity we adopt the following convention: p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\rm T}$, P_1, P_2 will be subsets of $\mathcal{E}^2_{\rm T}$, f_1, f_2 will be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r, s will be real numbers, n will be a natural number, and G will be a Go-board. We now state several propositions:

(1) Given G, f_1, f_2. Suppose that
- (i) $1 \leq \mathrm{len}\, f_1$,
- (ii) $1 \leq \mathrm{len}\, f_2$,
- (iii) f_1 is a sequence which elements belong to G,
- (iv) f_2 is a sequence which elements belong to G,
- (v) $f_1(1) \in \mathrm{rng}\, \mathrm{Line}(G, 1)$,
- (vi) $f_1(\mathrm{len}\, f_1) \in \mathrm{rng}\, \mathrm{Line}(G, \mathrm{len}\, G)$,
- (vii) $f_2(1) \in \mathrm{rng}(G_{\square,1})$,
- (viii) $f_2(\mathrm{len}\, f_2) \in \mathrm{rng}(G_{\square,\mathrm{width}\, G})$.

Then $\mathrm{rng}\, f_1 \cap \mathrm{rng}\, f_2 \neq \emptyset$.

(2) Given G, f_1, f_2. Suppose that
- (i) $2 \leq \mathrm{len}\, f_1$,
- (ii) $2 \leq \mathrm{len}\, f_2$,
- (iii) f_1 is a sequence which elements belong to G,
- (iv) f_2 is a sequence which elements belong to G,
- (v) $f_1(1) \in \mathrm{rng}\, \mathrm{Line}(G, 1)$,
- (vi) $f_1(\mathrm{len}\, f_1) \in \mathrm{rng}\, \mathrm{Line}(G, \mathrm{len}\, G)$,

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(vii) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(viii) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(3) Given G, f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) f_2 is a special sequence,
(iii) f_1 is a sequence which elements belong to G,
(iv) f_2 is a sequence which elements belong to G,
(v) $f_1(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$,
(vi) $f_1(\operatorname{len} f_1) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,
(vii) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(viii) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(4) Given f_1, f_2. Suppose that
(i) $2 \leq \operatorname{len} f_1$,
(ii) $2 \leq \operatorname{len} f_2$,
(iii) for all n, p, q such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ and $f_1(n) = p$ and $f_1(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,
(iv) for all n, p, q such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ and $f_2(n) = p$ and $f_2(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,
(v) for every n such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ holds $f_1(n) \neq f_1(n+1)$,
(vi) for every n such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ holds $f_2(n) \neq f_2(n+1)$,
(vii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(viii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(ix) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(x) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,
(xi) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(xii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(xiii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(xiv) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(5) Given f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) f_2 is a special sequence,

(iii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(iv) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(v) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(vi) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,
(vii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(viii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(ix) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(x) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.

Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(6) Given P_1, P_2. Suppose P_1 is a special polygonal arc and P_2 is a special polygonal arc. Given G, f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) $P_1 = \widetilde{\mathcal{L}}(f_1)$,
(iii) f_2 is a special sequence,
(iv) $P_2 = \widetilde{\mathcal{L}}(f_2)$,
(v) f_1 is a sequence which elements belong to G,
(vi) f_2 is a sequence which elements belong to G,
(vii) $f_1(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$,
(viii) $f_1(\operatorname{len} f_1) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,
(ix) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(x) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.

Then $P_1 \cap P_2 \neq \emptyset$.

(7) Given P_1, P_2. Suppose P_1 is a special polygonal arc and P_2 is a special polygonal arc. Given f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) $P_1 = \widetilde{\mathcal{L}}(f_1)$,
(iii) f_2 is a special sequence,
(iv) $P_2 = \widetilde{\mathcal{L}}(f_2)$,
(v) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(vi) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(vii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(viii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,

(ix) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,

(x) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,

(xi) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,

(xii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.

Then $P_1 \cap P_2 \neq \emptyset$.

(8) Given P_1, P_2, p_1, p_2, q_1, q_2. Suppose that

(i) P_1 is a special polygonal arc joining p_1 and q_1,

(ii) P_2 is a special polygonal arc joining p_2 and q_2,

(iii) for every p such that $p \in P_1 \cup P_2$ holds $p_{1\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq q_{1\mathbf{1}}$,

(iv) for every p such that $p \in P_1 \cup P_2$ holds $p_{2\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq q_{2\mathbf{2}}$.

Then $P_1 \cap P_2 \neq \emptyset$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[6] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[9] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.

[10] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[11] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part II. *Formalized Mathematics*, 3(**1**):117–121, 1992.

[12] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Formalized Mathematics*, 3(**1**):101–106, 1992.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[15] Yukio Takeuchi and Yatsuka Nakamura. *On the Jordan curve theorem.* Technical Report 19804, Dept. of Information Eng., Shinshu University, 500 Wakasato, Nagano city, Japan, April 1980.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

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