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Some Isomorphisms Between Functor Categories

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Summary. We define some well known isomorphisms between functor categories: between $A^{\dot{\bigcirc}(o,m)}$ and A, between $C^{[:A,B:]}$ and $(C^B)^A$, and between $[:B,C:]^A$ and $[:B^A,C^A:]$. Compare [12] and [11]. Unfortunately in this paper "functor" is used in two different meanings, as a lingual function and as a functor between categories.

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The notation and terminology used in this paper are introduced in the following papers: [17], [18], [4], [5], [3], [7], [1], [2], [10], [13], [8], [14], [6], [9], [16], and [15].

1. Preliminaries

The scheme *ChoiceD* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function h from $\mathcal{A}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds $\mathcal{P}[a, h(a)]$

provided the parameters meet the following requirement:

- for every element a of $\mathcal{A}$ there exists an element b of $\mathcal{B}$ such that $\mathcal{P}[a, b]$.

Let A, B, C be non-empty sets, and let f be a function from A into C^B. Then uncurry f is a function from $[:A, B:]$ into C.

We now state several propositions:

(1) For all non-empty sets A, B, C and for every function f from A into C^B holds curry uncurry $f = f$.

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(2) For all non-empty sets A, B, C and for every function f from A into C^B and for every element a of A and for every element b of B holds $(\mathrm{uncurry}\, f)(\langle a,\, b\rangle) = f(a)(b)$.

(3) For an arbitrary x and for every non-empty set A and for all functions f, g from $\{x\}$ into A such that $f(x) = g(x)$ holds $f = g$.

(4) For all non-empty sets A, B and for every element x of A and for every function f from A into B holds $f(x) \in \mathrm{rng}\, f$.

(5) For all non-empty sets A, B, C and for all functions f, g from A into $[\!:B, C\,:\!]$ such that $\pi_1(B\times C)\cdot f = \pi_1(B\times C)\cdot g$ and $\pi_2(B\times C)\cdot f = \pi_2(B\times C)\cdot g$ holds $f = g$.

We adopt the following rules: A, B, C will be categories and F, F_1, F_2 will be functors from A to B. The following two propositions are true:

(6) For every morphism f of A holds $\mathrm{id}_{\mathrm{cod}\, f} \cdot f = f$.

(7) For every morphism f of A holds $f \cdot \mathrm{id}_{\mathrm{dom}\, f} = f$.

In the sequel o, m will be arbitrary. The following two propositions are true:

(8) o is an object of B^A if and only if o is a functor from A to B.

(9) For every morphism f of B^A there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 and $\mathrm{dom}\, f = F_1$ and $\mathrm{cod}\, f = F_2$ and $f = \langle\langle F_1,\, F_2\rangle,\, t\rangle$.

2. The isomorphism between $A^{\dot{\circlearrowleft}(o,m)}$ and A

Let us consider A, B, and let a be an object of A. The functor $a \mapsto B$ yields a functor from B^A to B and is defined by:

(Def.1) for all functors F_1, F_2 from A to B and for every natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 holds $(a \mapsto B)(\langle\langle F_1,\, F_2\rangle,\, t\rangle) = t(a)$.

One can prove the following two propositions:

(10) The objects of $\dot{\circlearrowleft}(o,m) = \{o\}$ and the morphisms of $\dot{\circlearrowleft}(o,m) = \{m\}$.

(11) $A^{\dot{\circlearrowleft}(o,m)} \cong A$.

3. The isomorphism between $C^{[\!:A,B:\!]}$ and $(C^B)^A$

Next we state four propositions:

(12) For every functor F from $[\!:A, B\,:\!]$ to C and for every object a of A and for every object b of B holds $F(a,-)(b) = F(\langle a,\, b\rangle)$.

(13) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B holds $\mathrm{hom}(a_1, a_2) \neq \emptyset$ and $\mathrm{hom}(b_1, b_2) \neq \emptyset$ if and only if $\mathrm{hom}(\langle a_1,\, b_1\rangle, \langle a_2,\, b_2\rangle) \neq \emptyset$.

(14) Let a_1, a_2 be objects of A. Then for all objects b_1, b_2 of B such that $\hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$ and for every morphism f of A and for every morphism g of B holds $\langle f, g\rangle$ is a morphism from $\langle a_1, b_1\rangle$ to $\langle a_2, b_2\rangle$ if and only if f is a morphism from a_1 to a_2 and g is a morphism from b_1 to b_2.

(15) For all functors F_1, F_2 from $[\!:A, B:\!]$ to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 and for every object a of A holds $F_1(a, -)$ is naturally transformable to $F_2(a, -)$ and $(\text{curry}\, t)(a)$ is a natural transformation from $F_1(a, -)$ to $F_2(a, -)$.

Let us consider A, B, C, and let F be a functor from $[\!:A, B:\!]$ to C, and let f be a morphism of A. The functor $\text{curry}(F, f)$ yields a function from the morphisms of B into the morphisms of C and is defined by:

(Def.2) $\text{curry}(F, f) = (\text{curry}\, F)(f)$.

The following two propositions are true:

(16) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B and for every morphism f of A and for every morphism g of B such that $f \in \hom(a_1, a_2)$ and $g \in \hom(b_1, b_2)$ holds $\langle f, g\rangle \in \hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle)$.

(17) For every functor F from $[\!:A, B:\!]$ to C and for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ and for every morphism f from a to b holds $F(a, -)$ is naturally transformable to $F(b, -)$ and $\text{curry}(F, f) \cdot$ the id-map of B is a natural transformation from $F(a, -)$ to $F(b, -)$.

Let us consider A, B, C, and let F be a functor from $[\!:A, B:\!]$ to C, and let f be a morphism of A. The functor $F(f, -)$ yielding a natural transformation from $F(\text{dom}\, f, -)$ to $F(\text{cod}\, f, -)$ is defined by:

(Def.3) $F(f, -) = \text{curry}(F, f) \cdot$ the id-map of B.

We now state four propositions:

(18) For every functor F from $[\!:A, B:\!]$ to C and for every morphism g of A holds $F(\text{dom}\, g, -)$ is naturally transformable to $F(\text{cod}\, g, -)$.

(19) For every functor F from $[\!:A, B:\!]$ to C and for every morphism f of A and for every object b of B holds $F(f, -)(b) = F(\langle f, \text{id}_b\rangle)$.

(20) For every functor F from $[\!:A, B:\!]$ to C and for every object a of A holds $\text{id}_{F(a,-)} = F(\text{id}_a, -)$.

(21) For every functor F from $[\!:A, B:\!]$ to C and for all morphisms g, f of A such that $\text{dom}\, g = \text{cod}\, f$ and for every natural transformation t from $F(\text{dom}\, f, -)$ to $F(\text{dom}\, g, -)$ such that $t = F(f, -)$ holds $F(g \cdot f, -) = F(g, -) \circ t$.

Let us consider A, B, C, and let F be a functor from $[\!:A, B:\!]$ to C. The functor $\text{export}(F)$ yielding a functor from A to C^B is defined as follows:

(Def.4) for every morphism f of A holds $(\text{export}(F))(f) = \langle\langle F(\text{dom}\, f, -), F(\text{cod}\, f, -)\rangle, F(f, -)\rangle$.

We now state several propositions:

(22) For every functor F from $[: A, B :]$ to C and for every morphism f of A holds $(\mathrm{export}(F))(f) = \langle\langle F(\mathrm{dom}\, f, -), F(\mathrm{cod}\, f, -)\rangle, F(f, -)\rangle$.

(23) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every object a of A holds $t(a) \in \hom(F_1(a), F_2(a))$.

(24) For every functor F from $[: A, B :]$ to C and for every object a of A holds $(\mathrm{export}(F))(a) = F(a, -)$.

(25) For every functor F from $[: A, B :]$ to C and for every object a of A holds $(\mathrm{export}(F))(a)$ is a functor from B to C.

(26) For all functors F_1, F_2 from $[: A, B :]$ to C such that $\mathrm{export}(F_1) = \mathrm{export}(F_2)$ holds $F_1 = F_2$.

(27) Let F_1, F_2 be functors from $[: A, B :]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathrm{export}(F_1)$ is naturally transformable to $\mathrm{export}(F_2)$ and there exists a natural transformation G from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ such that for every function s from $[:$ the objects of A, the objects of $B :]$ into the morphisms of C such that $t = s$ and for every object a of A holds $G(a) = \langle\langle(\mathrm{export}(F_1))(a), (\mathrm{export}(F_2))(a)\rangle, (\mathrm{curry}\, s)(a)\rangle$.

Let us consider A, B, C, and let F_1, F_2 be functors from $[: A, B :]$ to C satisfying the condition: F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. The functor $\mathrm{export}(t)$ yielding a natural transformation from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ is defined as follows:

(Def.5) for every function s from $[:$ the objects of A, the objects of $B :]$ into the morphisms of C such that $t = s$ and for every object a of A holds $(\mathrm{export}(t))(a) = \langle\langle(\mathrm{export}(F_1))(a), (\mathrm{export}(F_2))(a)\rangle, (\mathrm{curry}\, s)(a)\rangle$.

We now state several propositions:

(28) For every functor F from $[: A, B :]$ to C holds $\mathrm{id}_{\mathrm{export}(F)} = \mathrm{export}(\mathrm{id}_F)$.

(29) For all functors F_1, F_2, F_3 from $[: A, B :]$ to C such that F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3 and for every natural transformation t_1 from F_1 to F_2 and for every natural transformation t_2 from F_2 to F_3 holds $\mathrm{export}(t_2 \circ t_1) = \mathrm{export}(t_2) \circ \mathrm{export}(t_1)$.

(30) For all functors F_1, F_2 from $[: A, B :]$ to C such that F_1 is naturally transformable to F_2 and for all natural transformations t_1, t_2 from F_1 to F_2 such that $\mathrm{export}(t_1) = \mathrm{export}(t_2)$ holds $t_1 = t_2$.

(31) For every functor G from A to C^B there exists a functor F from $[: A, B :]$ to C such that $G = \mathrm{export}(F)$.

(32) For all functors F_1, F_2 from $[: A, B :]$ to C such that $\mathrm{export}(F_1)$ is naturally transformable to $\mathrm{export}(F_2)$ and for every natural transformation t from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ holds F_1 is naturally transformable to F_2 and there exists a natural transformation u from F_1 to F_2 such that $t = \mathrm{export}(u)$.

Let us consider A, B, C. The functor $\mathbf{export}_{A,B,C}$ yields a functor from $C^{[: A, B :]}$ to $(C^B)^A$ and is defined by:

(Def.6) for all functors F_1, F_2 from $[: A, B :]$ to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $\mathbf{export}_{A,B,C}(\langle\langle F_1, F_2\rangle, t\rangle) = \langle\langle \mathrm{export}(F_1), \mathrm{export}(F_2)\rangle, \mathrm{export}(t)\rangle$.

Next we state two propositions:

(33) $\mathbf{export}_{A,B,C}$ is an isomorphism.

(34) $C^{[: A, B :]} \cong (C^B)^A$.

4. The isomorphism between $[:B, C:]^A$ and $[:B^A, C^A:]$

We now state the proposition

(35) For all functors F_1, F_2 from A to B and for every functor G from B to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $G \cdot t = G \cdot t$ **qua** a function .

We now define two new functors. Let us consider A, B. Then $\pi_1(A \times B)$ is a functor from $[: A, B :]$ to A. Then $\pi_2(A \times B)$ is a functor from $[: A, B :]$ to B. Let us consider A, B, C, and let F be a functor from A to B, and let G be a functor from A to C. Then $\langle F, G\rangle$ is a functor from A to $[: B, C :]$. Let F be a functor from A to $[: B, C :]$. The functor $\pi_1 \cdot F$ yielding a functor from A to B is defined as follows:

(Def.7) $\pi_1 \cdot F = \pi_1(B \times C) \cdot F$.

The functor $\pi_2 \cdot F$ yielding a functor from A to C is defined by:

(Def.8) $\pi_2 \cdot F = \pi_2(B \times C) \cdot F$.

The following two propositions are true:

(36) For every functor F from A to B and for every functor G from A to C holds $\pi_1 \cdot \langle F, G\rangle = F$ and $\pi_2 \cdot \langle F, G\rangle = G$.

(37) For all functors F, G from A to $[: B, C :]$ such that $\pi_1 \cdot F = \pi_1 \cdot G$ and $\pi_2 \cdot F = \pi_2 \cdot G$ holds $F = G$.

We now define two new functors. Let us consider A, B, C, and let F_1, F_2 be functors from A to $[: B, C :]$, and let t be a natural transformation from F_1 to F_2. The functor $\pi_1 \cdot t$ yielding a natural transformation from $\pi_1 \cdot F_1$ to $\pi_1 \cdot F_2$ is defined as follows:

(Def.9) $\pi_1 \cdot t = \pi_1(B \times C) \cdot t$.

The functor $\pi_2 \cdot t$ yielding a natural transformation from $\pi_2 \cdot F_1$ to $\pi_2 \cdot F_2$ is defined as follows:

(Def.10) $\pi_2 \cdot t = \pi_2(B \times C) \cdot t$.

We now state several propositions:

(38) For all functors F, G from A to $[\!: B, C :\!]$ such that F is naturally transformable to G holds $\pi_1 \cdot F$ is naturally transformable to $\pi_1 \cdot G$ and $\pi_2 \cdot F$ is naturally transformable to $\pi_2 \cdot G$.

(39) For all functors F_1, F_2, G_1, G_2 from A to $[\!: B, C :\!]$ such that F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 and for every natural transformation s from F_1 to F_2 and for every natural transformation t from G_1 to G_2 such that $\pi_1 \cdot s = \pi_1 \cdot t$ and $\pi_2 \cdot s = \pi_2 \cdot t$ holds $s = t$.

(40) For every functor F from A to $[\!: B, C :\!]$ holds $\mathrm{id}_{\pi_1 \cdot F} = \pi_1 \cdot (\mathrm{id}_F)$ and $\mathrm{id}_{\pi_2 \cdot F} = \pi_2 \cdot (\mathrm{id}_F)$.

(41) For all functors F, G, H from A to $[\!: B, C :\!]$ such that F is naturally transformable to G and G is naturally transformable to H and for every natural transformation s from F to G and for every natural transformation t from G to H holds $\pi_1 \cdot (t \circ s) = \pi_1 \cdot t \circ \pi_1 \cdot s$ and $\pi_2 \cdot (t \circ s) = \pi_2 \cdot t \circ \pi_2 \cdot s$.

(42) For every functor F from A to B and for every functor G from A to C and for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ and for every morphism f from a to b holds $\langle F, G \rangle(f) = \langle\!\langle F(f), G(f) \rangle\!\rangle$.

(43) For every functor F from A to B and for every functor G from A to C and for every object a of A holds $\langle F, G \rangle(a) = \langle\!\langle F(a), G(a) \rangle\!\rangle$.

(44) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is transformable to G_1 and F_2 is transformable to G_2 holds $\langle F_1, F_2 \rangle$ is transformable to $\langle G_1, G_2 \rangle$.

Let us consider A, B, C, and let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C satisfying the condition: F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1, and let t_2 be a transformation from F_2 to G_2. The functor $\langle t_1, t_2 \rangle$ yielding a transformation from $\langle F_1, F_2 \rangle$ to $\langle G_1, G_2 \rangle$ is defined as follows:

(Def.11) $\langle t_1, t_2 \rangle = \langle t_1, t_2 \rangle$.

One can prove the following propositions:

(45) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is transformable to G_1 and F_2 is transformable to G_2 and for every transformation t_1 from F_1 to G_1 and for every transformation t_2 from F_2 to G_2 and for every object a of A holds $\langle t_1, t_2 \rangle(a) = \langle\!\langle t_1(a), t_2(a) \rangle\!\rangle$.

(46) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2 holds $\langle F_1, F_2 \rangle$ is naturally transformable to $\langle G_1, G_2 \rangle$.

Let us consider A, B, C, and let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C satisfying the conditions: F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1, and let t_2 be a natural transformation from F_2 to G_2. The functor $\langle t_1, t_2 \rangle$ yielding a natural transformation from $\langle F_1, F_2 \rangle$ to $\langle G_1, G_2 \rangle$ is defined as follows:

(Def.12) $\langle t_1, t_2 \rangle = \langle t_1, t_2 \rangle$.

Next we state the proposition

(47) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2 and for every natural transformation t_1 from F_1 to G_1 and for every natural transformation t_2 from F_2 to G_2 holds $\pi_1 \cdot \langle t_1, t_2 \rangle = t_1$ and $\pi_2 \cdot \langle t_1, t_2 \rangle = t_2$.

Let us consider A, B, C. The functor $\mathbf{distribute}_{A,B,C}$ yielding a functor from $[\!: B, C :\!]^A$ to $[\!: B^A, C^A :\!]$ is defined by:

(Def.13) for all functors F_1, F_2 from A to $[\!: B, C :\!]$ such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $\mathbf{distribute}_{A,B,C}(\langle\langle F_1, F_2 \rangle, t\rangle) = \langle\langle\langle \pi_1 \cdot F_1, \pi_1 \cdot F_2 \rangle, \pi_1 \cdot t\rangle, \langle\langle \pi_2 \cdot F_1, \pi_2 \cdot F_2 \rangle, \pi_2 \cdot t\rangle\rangle$.

One can prove the following two propositions:

(48) $\mathbf{distribute}_{A,B,C}$ is an isomorphism.

(49) $[\!: B, C :\!]^A \cong [\!: B^A, C^A :\!]$.

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