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On Paracompactness of Metrizable Spaces

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Summary. The aim is to prove, using Mizar System, one of the most important result in general topology, namely the Stone Theorem on paracompactness of metrizable spaces [19]. Our proof is based on [18] (and also [16]). We prove first auxiliary fact that every open cover of any metrizable space has a locally finite open refinement. We show next the main theorem that every metrizable space is paracompact. The remaining material is devoted to concepts and certain properties needed for the formulation and the proof of that theorem (see also [5]).

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The notation and terminology used here are introduced in the following articles: [21], [7], [8], [13], [26], [15], [10], [20], [11], [23], [1], [14], [9], [5], [12], [17], [24], [2], [3], [4], [25], [6], and [22].

1. Selected Properties of Real Numbers

We adopt the following rules: r, u, v, w, y are real numbers and k is a natural number. One can prove the following propositions:

(1) $r_{\mathbb{N}}^{0} = 1$.

(2) $r_{\mathbb{N}}^{1} = r$.

(3) If $r > 0$ and $u > 0$, then there exists a natural number k such that $\frac{u}{2_{\mathbb{N}}^{k}} \leq r$.

(4) If $k \geq n$ and $r \geq 1$, then $r_{\mathbb{N}}^{k} \geq r_{\mathbb{N}}^{n}$.

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2. Certain Functions Defined on Families of Sets

We adopt the following convention: R will be a binary relation, A, B, C will be sets, and t will be arbitrary. The following proposition is true

(5) If R well orders A, then $R\,|^2\,A$ well orders A and $A = \mathrm{field}(R\,|^2\,A)$.

The scheme *MinSet* concerns a set $\mathcal{A}$, a binary relation $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists arbitrary X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$ and for an arbitrary Y such that $Y \in \mathcal{A}$ and $\mathcal{P}[Y]$ holds $\langle X, Y \rangle \in \mathcal{B}$

provided the parameters meet the following conditions:

- $\mathcal{B}$ well orders $\mathcal{A}$,
- there exists arbitrary X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$.

We now define three new functors. Let F_1 be a family of sets, and let R be a binary relation, and let B be an element of F_1. The functor $\bigcup_{\beta <_R B} \beta$ yields a family of sets and is defined as follows:

(Def.1) $\bigcup_{\beta <_R B} \beta = \bigcup(R\text{–}\mathrm{Seg}(B))$.

Let F_1 be a family of sets, and let R be a binary relation. The disjoint family of F_1, R yielding a family of sets is defined by:

(Def.2) $A \in$ the disjoint family of F_1, R if and only if there exists an element B of F_1 such that $B \in F_1$ and $A = B \setminus \bigcup_{\beta <_R B} \beta$.

Let X be a set, and let n be a natural number, and let f be a function from $\mathbb{N}$ into 2^X. The functor $\bigcup_{\kappa < n} f(\kappa)$ yields a set and is defined as follows:

(Def.3) $\bigcup_{\kappa < n} f(\kappa) = \bigcup(f\,^\circ\,(\mathrm{Seg}\,n \setminus \{n\}))$.

3. Paracompactness of Metrizable Spaces

We adopt the following convention: P_1 will denote a topological space, F_1, G_1 will denote families of subsets of P_1, and W, X will denote subsets of P_1. We now state several propositions:

(6) If P_1 is a T_3 space, then for every F_1 such that F_1 is a cover of P_1 and F_1 is open there exists H_1 such that H_1 is open and H_1 is a cover of P_1 and for every V such that $V \in H_1$ there exists W such that $W \in F_1$ and $\overline{V} \subseteq W$.

(7) For all P_1, F_1 such that P_1 is a T_2 space and P_1 is paracompact and F_1 is a cover of P_1 and F_1 is open there exists G_1 such that G_1 is open and G_1 is a cover of P_1 and $\mathrm{clf}\,G_1$ is finer than F_1 and G_1 is locally finite.

(8) For every function f from $[\!:$ the carrier of P_1, the carrier of $P_1 :\!]$ into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the carrier of $P_1), f)$, then the carrier of $P_2 =$ the carrier of P_1.

(9) For every function f from $[\!:$ the carrier of P_1, the carrier of $P_1 :\!]$ into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the

carrier of P_1), f), then x is a point of P_1 if and only if x is an element of the carrier of P_2.

(10) For every function f from [: the carrier of P_1, the carrier of P_1 :] into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if P_2 = MetrSp((the carrier of P_1), f), then X is a subset of P_1 if and only if X is a subset of the carrier of P_2.

(11) For every function f from [: the carrier of P_1, the carrier of P_1 :] into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if P_2 = MetrSp((the carrier of P_1), f), then F_1 is a family of subsets of P_1 if and only if F_1 is a family of subsets of the carrier of P_2.

In the sequel k is a natural number. Let P_2 be a non-empty set, and let g be a function from $\mathbb{N}$ into $(2^{2^{P_2}})^*$, and let us consider n. Then $g(n)$ is a finite sequence of elements of $2^{2^{P_2}}$.

The following propositions are true:

(12) If P_1 is metrizable, then for every family F_1 of subsets of P_1 such that F_1 is a cover of P_1 and F_1 is open there exists a family G_1 of subsets of P_1 such that G_1 is open and G_1 is a cover of P_1 and G_1 is finer than F_1 and G_1 is locally finite.

(13) If P_1 is metrizable, then P_1 is paracompact.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[3] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[10] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[16] Hanna Patkowska. *Wstęp do Topologii.* PWN, Warszawa, 1974.

[17] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.
[18] M. E. Rudin. A new proof that metric spaces are paracompact. *Proc. Amer. Math. Soc.*, 20:603, 1969.
[19] A. H. Stone. Paracompactness and product spaces. *Bull. Amer. Math. Soc.*, 54:977–982, 1948.
[20] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.
[26] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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