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The Lattice of Domains of a Topological Space [1]

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Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *closed domain* of T if $A = \overline{\operatorname{Int} A}$ and A is said to be an *open domain* of T if $A = \operatorname{Int} \overline{A}$ (see e.g. [8], [15]). Some simple generalization of these notions is the following one. A is said to be a *domain* of T provided $\operatorname{Int} \overline{A} \subseteq A \subseteq \overline{\operatorname{Int} A}$ (see [15] and compare [7]). In this paper certain connections between these concepts are introduced and studied.

Our main results are concerned with the following well–known theorems (see e.g. [9], [2]). For a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too. It is proved that *all domains of a given topological space form a complemented lattice.* Moreover, it is shown that both *the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains.* In the beginning some useful theorems about subsets of topological spaces are proved and certain properties of domains, closed domains and open domains are discussed.

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The terminology and notation used in this paper are introduced in the following articles: [14], [11], [4], [5], [16], [3], [13], [10], [15], [1], [12], and [6].

1. Preliminary Theorems on Subset of Topological Spaces

In the sequel T is a topological space. We now state a number of propositions:

(1) For all subsets A, B of T holds $A \cup B = \Omega_T$ if and only if $A^{\rm c} \subseteq B$.

(2) For all subsets A, B of T holds $A \cap B = \emptyset_T$ if and only if $B \subseteq A^{\rm c}$.

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(3) For every subset A of T holds $\overline{\operatorname{Int} \overline{A}} \subseteq \overline{A}$.

(4) For every subset A of T holds $\operatorname{Int} A \subseteq \operatorname{Int} \overline{\operatorname{Int} A}$.

(5) For every subset A of T holds $\operatorname{Int} \overline{A} = \operatorname{Int} \overline{\operatorname{Int} \overline{A}}$.

(6) For all subsets A, B of T such that A is closed or B is closed holds $\overline{\operatorname{Int} A} \cup \overline{\operatorname{Int} B} = \overline{\operatorname{Int}(A \cup B)}$.

(7) For all subsets A, B of T such that A is open or B is open holds $\operatorname{Int} \overline{A} \cap \operatorname{Int} \overline{B} = \operatorname{Int} \overline{A \cap B}$.

(8) For every subset A of T holds $\operatorname{Int}(A \cap \overline{A^{\rm c}}) = \emptyset_T$.

(9) For every subset A of T holds $\overline{A \cup \operatorname{Int}(A^{\rm c})} = \Omega_T$.

(10) For all subsets A, B of T holds $\operatorname{Int} \overline{A \cup (\operatorname{Int} \overline{B} \cup B)} \cup (A \cup (\operatorname{Int} \overline{B} \cup B)) = \operatorname{Int} \overline{A \cup B} \cup (A \cup B)$.

(11) For all subsets A, C of T holds $\operatorname{Int} \overline{\operatorname{Int} \overline{A} \cup A \cup C} \cup (\operatorname{Int} \overline{A} \cup A \cup C) = \operatorname{Int} \overline{A \cup C} \cup (A \cup C)$.

(12) For all subsets A, B of T holds $\overline{\operatorname{Int}(A \cap (\overline{\operatorname{Int} B} \cap B))} \cap (A \cap (\overline{\operatorname{Int} B} \cap B)) = \overline{\operatorname{Int}(A \cap B)} \cap (A \cap B)$.

(13) For all subsets A, C of T holds $\overline{\operatorname{Int}(\overline{\operatorname{Int} A} \cap A \cap C)} \cap (\overline{\operatorname{Int} A} \cap A \cap C) = \overline{\operatorname{Int}(A \cap C)} \cap (A \cap C)$.

2. Properties of Domains of Topological Spaces

In the sequel T will be a topological space. Next we state a number of propositions:

(14) $\emptyset_T$ is a domain.

(15) Ω_T is a domain.

(16) For every subset A of T such that A is a domain holds $A^{\rm c}$ is a domain.

(17) For all subsets A, B of T such that A is a domain and B is a domain holds $\operatorname{Int} \overline{A \cup B} \cup (A \cup B)$ is a domain and $\overline{\operatorname{Int}(A \cap B)} \cap (A \cap B)$ is a domain.

(18) $\emptyset_T$ is a closed domain.

(19) Ω_T is a closed domain.

(20) $\emptyset_T$ is an open domain.

(21) Ω_T is an open domain.

(22) For every subset A of T holds $\overline{\operatorname{Int} A}$ is a closed domain.

(23) For every subset A of T holds $\operatorname{Int} \overline{A}$ is an open domain.

(24) For every subset A of T such that A is a domain holds $\overline{A}$ is a closed domain.

(25) For every subset A of T such that A is a domain holds $\operatorname{Int} A$ is an open domain.

(26) For every subset A of T such that A is a domain holds $\overline{A^{\rm c}}$ is a closed domain.

(27) For every subset A of T such that A is a domain holds $\mathrm{Int}(A^{\mathrm{c}})$ is an open domain.

(28) For all subsets A, B, C of T such that A is a closed domain and B is a closed domain and C is a closed domain holds $\overline{\mathrm{Int}(A \cap \overline{\mathrm{Int}(B \cap C)})} = \overline{\mathrm{Int}(\overline{\mathrm{Int}(A \cap B)} \cap C)}$.

(29) For all subsets A, B, C of T such that A is an open domain and B is an open domain and C is an open domain holds $\mathrm{Int}\,\overline{A \cup \mathrm{Int}\,\overline{B \cup C}} = \mathrm{Int}\,\overline{\mathrm{Int}\,\overline{A \cup B} \cup C}$.

3. The Lattice of Domains

We now define five new functors. Let T be a topological space. The domains of T yields a non-empty family of subsets of the carrier of T and is defined as follows:

(Def.1) the domains of $T = \{A : A$ is a domain$\}$, where A ranges over subsets of T.

The domains union of T yielding a binary operation on the domains of T is defined by:

(Def.2) for all elements A, B of the domains of T holds (the domains union of T)$(A, B) = \mathrm{Int}\,\overline{A \cup B} \cup (A \cup B)$.

We introduce the functor D-Union(T) as a synonym of the domains union of T. The domains meet of T yields a binary operation on the domains of T and is defined as follows:

(Def.3) for all elements A, B of the domains of T holds (the domains meet of T)$(A, B) = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

We introduce the functor D-Meet(T) as a synonym of the domains meet of T.

One can prove the following proposition

(30) For every topological space T holds $\langle$the domains of T, D-Union(T), D-Meet(T)$\rangle$ is a complemented lattice.

Let T be a topological space. The lattice of domains of T yields a complemented lattice and is defined by:

(Def.4) the lattice of domains of $T = \langle$the domains of T, the domains union of T, the domains meet of $T\rangle$.

4. The Lattice of Closed Domains

Let T be a topological space. The closed domains of T yielding a non-empty family of subsets of the carrier of T is defined as follows:

(Def.5) the closed domains of $T = \{A : A$ is a closed domain$\}$, where A ranges over subsets of T.

Next we state the proposition

(31) For every topological space T holds the closed domains of $T \subseteq$ the domains of T.

We now define two new functors. Let T be a topological space. The closed domains union of T yielding a binary operation on the closed domains of T is defined by:

(Def.6) for all elements A, B of the closed domains of T holds (the closed domains union of T)$(A, B) = A \cup B$.

We introduce the functor CLD-Union(T) as a synonym of the closed domains union of T.

Next we state the proposition

(32) For all elements A, B of the closed domains of T holds (CLD-Union(T))(A, B) = (D-Union(T))(A, B).

We now define two new functors. Let T be a topological space. The closed domains meet of T yielding a binary operation on the closed domains of T is defined as follows:

(Def.7) for all elements A, B of the closed domains of T holds (the closed domains meet of T)$(A, B) = \overline{\mathrm{Int}(A \cap B)}$.

We introduce the functor CLD-Meet(T) as a synonym of the closed domains meet of T.

One can prove the following two propositions:

(33) For all elements A, B of the closed domains of T holds (CLD-Meet(T))(A, B) = (D-Meet(T))(A, B).

(34) For every topological space T holds $\langle$the closed domains of T, CLD-Union(T), CLD-Meet(T)$\rangle$ is a Boolean lattice.

Let T be a topological space. The lattice of closed domains of T yielding a Boolean lattice is defined as follows:

(Def.8) the lattice of closed domains of $T = \langle$the closed domains of T, the closed domains union of T, the closed domains meet of $T\rangle$.

5. The Lattice of Open Domains

Let T be a topological space. The open domains of T yields a non-empty family of subsets of the carrier of T and is defined by:

(Def.9) the open domains of $T = \{A : A$ is an open domain$\}$, where A ranges over subsets of T.

Next we state the proposition

(35) For every topological space T holds the open domains of $T \subseteq$ the domains of T.

We now define two new functors. Let T be a topological space. The open domains union of T yielding a binary operation on the open domains of T is defined by:

(Def.10) for all elements A, B of the open domains of T holds (the open domains union of T)$(A, B) = \mathrm{Int}\,\overline{A \cup B}$.

We introduce the functor OPD-Union(T) as a synonym of the open domains union of T.

One can prove the following proposition

(36) For all elements A, B of the open domains of T holds (OPD-Union(T))(A, B) = (D-Union(T))(A, B).

We now define two new functors. Let T be a topological space. The open domains meet of T yielding a binary operation on the open domains of T is defined by:

(Def.11) for all elements A, B of the open domains of T holds (the open domains meet of T)$(A, B) = A \cap B$.

We introduce the functor OPD-Meet(T) as a synonym of the open domains meet of T.

We now state two propositions:

(37) For all elements A, B of the open domains of T holds (OPD-Meet(T))(A, B) = (D-Meet(T))(A, B).

(38) For every topological space T holds $\langle$the open domains of T, OPD-Union(T), OPD-Meet(T)$\rangle$ is a Boolean lattice.

Let T be a topological space. The lattice of open domains of T yielding a Boolean lattice is defined by:

(Def.12) the lattice of open domains of T = $\langle$the open domains of T, the open domains union of T, the open domains meet of $T\rangle$.

6. Connections between Lattices of Domains

In the sequel T will be a topological space. The following propositions are true:

(39) CLD-Union(T) = D-Union(T) $\restriction$ [: the closed domains of T, the closed domains of T :].

(40) CLD-Meet(T) = D-Meet(T) $\restriction$ [: the closed domains of T, the closed domains of T :].

(41) The lattice of closed domains of T is a sublattice of the lattice of domains of T.

(42) OPD-Union(T) = D-Union(T) $\restriction$ [: the open domains of T, the open domains of T :].

(43) OPD-Meet(T) = D-Meet(T) $\restriction$ [: the open domains of T, the open domains of T :].

(44) The lattice of open domains of T is a sublattice of the lattice of domains of T.

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References

[1] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.
[2] Garrett Birkhoff. *Lattice Theory*. Providence, Rhode Island, New York, 1967.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Formalized Mathematics*, 2(**4**):453–459, 1991.
[7] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.
[8] Kazimierz Kuratowski. *Topology*. Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[9] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory (with an introduction to descriptive set theory)*. Volume 86 of *Studies in Logic and The Foundations of Mathematics*, PWN - Polish Scientific Publishers and North-Holland Publishing Company, Warsaw-Amsterdam, 1976.
[10] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[15] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.
[16] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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