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Completeness of the Lattices of Domains of a Topological Space [1]

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Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *domain* in T provided $\operatorname{Int}\overline{A} \subseteq A \subseteq \overline{\operatorname{Int} A}$ (see [24] and comp. [14]). This notion is a simple generalization of the notions of open and closed domains in T (see [24]). Our main result is concerned with an extension of the following well–known theorem (see e.g. [5], [17], [13]). For a given topological space the Boolean lattices of all its closed domains and all its open domains are complete. It is proved here, using Mizar System, that *the complemented lattice of all domains of a given topological space is complete*, too (comp. [23]).

It is known that both the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains [23]. However, the following two problems remain open.

> **Problem 1.** Let L be a sublattice of the lattice of all domains. Suppose L is complete, is smallest with respect to inclusion, and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all domains ?

A domain in T is said to be a *Borel domain* provided it is a Borel set. Of course every open (closed) domain is a Borel domain. It can be proved that all Borel domains form a sublattice of the lattice of domains.

> **Problem 2.** Let L be a sublattice of the lattice of all domains. Suppose L is smallest with respect to inclusion and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all Borel domains ?

Note that in the beginning the closure and the interior operations for families of subsets of topological spaces are introduced and their important properties are presented (comp. [16], [15], [17]). Using these notions, certain properties of domains, closed domains and open domains are studied (comp. [15], [13]).

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The papers [20], [22], [21], [18], [8], [9], [12], [4], [3], [19], [24], [11], [6], [7], [25], [10], [2], [1], and [23] provide the notation and terminology for this paper.

1. Preliminary Theorems about Subsets of Topological Spaces

In the sequel T will denote a topological space. One can prove the following propositions:

(1) For every subset A of T holds $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \subseteq \mathrm{Int}\,\overline{A}$ and $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \subseteq \overline{\mathrm{Int}\,A}$.

(2) For every subset A of T holds $\overline{\mathrm{Int}\,A} \subseteq \overline{\mathrm{Int}\,\overline{A}}$ and $\mathrm{Int}\,\overline{A} \subseteq \overline{\mathrm{Int}\,\overline{A}}$.

(3) For all subsets A, B of T such that B is closed holds if $\overline{\mathrm{Int}(A \cap B)} = A$, then $A \subseteq B$.

(4) For all subsets A, B of T such that A is open holds if $\mathrm{Int}\,\overline{A \cup B} = B$, then $A \subseteq B$.

(5) For every subset A of T such that $A \subseteq \overline{\mathrm{Int}\,A}$ holds $A \cup \mathrm{Int}\,\overline{A} \subseteq \overline{\mathrm{Int}(A \cup \mathrm{Int}\,\overline{A})}$.

(6) For every subset A of T such that $\mathrm{Int}\,\overline{A} \subseteq A$ holds $\mathrm{Int}\,\overline{A \cap \overline{\mathrm{Int}\,A}} \subseteq A \cap \overline{\mathrm{Int}\,A}$.

2. The Closure and the Interior Operations for Families of Subsets of a Topological Space

In the sequel T will be a topological space. Let us consider T, and let F be a family of subsets of T. We introduce the functor $\overline{F}$ as a synonym of clf F.

One can prove the following propositions:

(7) For every family F of subsets of T holds $\overline{F} = \{A : \bigvee_B [A = \overline{B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(8) For every family F of subsets of T holds $\overline{F} = \overline{\overline{F}}$.

(9) For every family F of subsets of T holds $F = \emptyset$ if and only if $\overline{F} = \emptyset$.

(10) For all families F, G of subsets of T holds $\overline{F \cap G} \subseteq \overline{F} \cap \overline{G}$.

(11) For all families F, G of subsets of T holds $\overline{F} \setminus \overline{G} \subseteq \overline{F \setminus G}$.

(12) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\bigcap \overline{F} \subseteq \overline{A}$ and $\overline{A} \subseteq \bigcup \overline{F}$.

(13) For every family F of subsets of T holds $\bigcap F \subseteq \bigcap \overline{F}$.

(14) For every family F of subsets of T holds $\overline{\bigcap F} \subseteq \bigcap \overline{F}$.

(15) For every family F of subsets of T holds $\bigcup \overline{F} \subseteq \overline{\bigcup F}$.

Let us consider T, and let F be a family of subsets of T. The functor $\mathrm{Int}\,F$ yielding a family of subsets of T is defined as follows:

(Def.1) for every subset A of T holds $A \in \mathrm{Int}\,F$ if and only if there exists a subset B of T such that $A = \mathrm{Int}\,B$ and $B \in F$.

The following propositions are true:

(16) For every family F of subsets of T holds $\mathrm{Int}\, F = \{A : \bigvee_B [A = \mathrm{Int}\, B \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(17) For every family F of subsets of T holds $\mathrm{Int}\, F = \mathrm{Int}\, \mathrm{Int}\, F$.

(18) For every family F of subsets of T holds $\mathrm{Int}\, F$ is open.

(19) For every family F of subsets of T holds $F = \emptyset$ if and only if $\mathrm{Int}\, F = \emptyset$.

(20) For every subset A of T and for every family F of subsets of T such that $F = \{A\}$ holds $\mathrm{Int}\, F = \{\mathrm{Int}\, A\}$.

(21) For all families F, G of subsets of T such that $F \subseteq G$ holds $\mathrm{Int}\, F \subseteq \mathrm{Int}\, G$.

(22) For all families F, G of subsets of T holds $\mathrm{Int}(F \cup G) = \mathrm{Int}\, F \cup \mathrm{Int}\, G$.

(23) For all families F, G of subsets of T holds $\mathrm{Int}(F \cap G) \subseteq \mathrm{Int}\, F \cap \mathrm{Int}\, G$.

(24) For all families F, G of subsets of T holds $\mathrm{Int}\, F \setminus \mathrm{Int}\, G \subseteq \mathrm{Int}(F \setminus G)$.

(25) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\mathrm{Int}\, A \subseteq \bigcup \mathrm{Int}\, F$ and $\bigcap \mathrm{Int}\, F \subseteq \mathrm{Int}\, A$.

(26) For every family F of subsets of T holds $\bigcup \mathrm{Int}\, F \subseteq \bigcup F$.

(27) For every family F of subsets of T holds $\bigcap \mathrm{Int}\, F \subseteq \bigcap F$.

(28) For every family F of subsets of T holds $\bigcup \mathrm{Int}\, F \subseteq \mathrm{Int} \bigcup F$.

(29) For every family F of subsets of T holds $\mathrm{Int} \bigcap F \subseteq \bigcap \mathrm{Int}\, F$.

(30) For every family F of subsets of T such that F is finite holds $\mathrm{Int} \bigcap F = \bigcap \mathrm{Int}\, F$.

In the sequel F denotes a family of subsets of T. The following propositions are true:

(31) $\overline{\mathrm{Int}\, F} = \{A : \bigvee_B [A = \overline{\mathrm{Int}\, B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(32) $\mathrm{Int}\, \overline{F} = \{A : \bigvee_B [A = \mathrm{Int}\, \overline{B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(33) $\overline{\mathrm{Int}\, \overline{F}} = \{A : \bigvee_B [A = \overline{\mathrm{Int}\, \overline{B}} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(34) $\mathrm{Int}\, \overline{\mathrm{Int}\, F} = \{A : \bigvee_B [A = \mathrm{Int}\, \overline{\mathrm{Int}\, B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(35) $\overline{\mathrm{Int}\, \overline{\mathrm{Int}\, F}} = \overline{\mathrm{Int}\, F}$.

(36) $\mathrm{Int}\, \overline{\mathrm{Int}\, \overline{F}} = \mathrm{Int}\, \overline{F}$.

(37) $\bigcup \mathrm{Int}\, \overline{F} \subseteq \bigcup \overline{\mathrm{Int}\, \overline{F}}$.

(38) $\bigcap \mathrm{Int}\, \overline{F} \subseteq \bigcap \overline{\mathrm{Int}\, \overline{F}}$.

(39) $\bigcup \overline{\mathrm{Int}\, F} \subseteq \bigcup \overline{\mathrm{Int}\, \overline{F}}$.

(40) $\bigcap \overline{\mathrm{Int}\, F} \subseteq \bigcap \overline{\mathrm{Int}\, \overline{F}}$.

(41) $\bigcup \mathrm{Int}\, \overline{\mathrm{Int}\, F} \subseteq \bigcup \mathrm{Int}\, \overline{F}$.

(42) $\bigcap \operatorname{Int} \overline{\operatorname{Int} F} \subseteq \bigcap \operatorname{Int} \overline{F}$.

(43) $\bigcup \operatorname{Int} \overline{\operatorname{Int} F} \subseteq \bigcup \overline{\operatorname{Int} F}$.

(44) $\bigcap \operatorname{Int} \overline{\operatorname{Int} F} \subseteq \bigcap \overline{\operatorname{Int} F}$.

(45) $\bigcup \overline{\operatorname{Int} \overline{F}} \subseteq \bigcup \overline{F}$.

(46) $\bigcap \overline{\operatorname{Int} \overline{F}} \subseteq \bigcap \overline{F}$.

(47) $\bigcup \operatorname{Int} F \subseteq \bigcup \operatorname{Int} \overline{\operatorname{Int} F}$.

(48) $\bigcap \operatorname{Int} F \subseteq \bigcap \operatorname{Int} \overline{\operatorname{Int} F}$.

(49) $\bigcup \overline{\operatorname{Int} F} \subseteq \overline{\operatorname{Int} \bigcup F}$.

(50) $\overline{\operatorname{Int} \bigcap F} \subseteq \bigcap \overline{\operatorname{Int} F}$.

(51) $\bigcup \operatorname{Int} \overline{F} \subseteq \operatorname{Int} \overline{\bigcup F}$.

(52) $\operatorname{Int} \overline{\bigcap F} \subseteq \bigcap \operatorname{Int} \overline{F}$.

(53) $\bigcup \overline{\operatorname{Int} \overline{F}} \subseteq \overline{\operatorname{Int} \overline{\bigcup F}}$.

(54) $\overline{\operatorname{Int} \overline{\bigcap F}} \subseteq \bigcap \overline{\operatorname{Int} \overline{F}}$.

(55) $\bigcup \operatorname{Int} \overline{\operatorname{Int} F} \subseteq \operatorname{Int} \overline{\operatorname{Int} \bigcup F}$.

(56) $\operatorname{Int} \overline{\operatorname{Int} \bigcap F} \subseteq \bigcap \operatorname{Int} \overline{\operatorname{Int} F}$.

(57) For every family F of subsets of T such that for every subset A of T such that $A \in F$ holds $A \subseteq \overline{\operatorname{Int} A}$ holds $\bigcup F \subseteq \overline{\operatorname{Int} \bigcup F}$ and $\overline{\bigcup F} = \overline{\operatorname{Int} \overline{\bigcup F}}$.

(58) For every family F of subsets of T such that for every subset A of T such that $A \in F$ holds $\operatorname{Int} \overline{A} \subseteq A$ holds $\operatorname{Int} \overline{\bigcap F} \subseteq \bigcap F$ and $\operatorname{Int} \overline{\operatorname{Int} \bigcap F} = \operatorname{Int} \bigcap F$.

3. Selected Properties of Domains of a Topological Space

In the sequel T is a topological space. We now state several propositions:

(59) For all subsets A, B of T such that B is a domain holds $\operatorname{Int} \overline{A \cup B} \cup (A \cup B) = B$ if and only if $A \subseteq B$.

(60) For all subsets A, B of T such that A is a domain holds $\overline{\operatorname{Int}(A \cap B)} \cap (A \cap B) = A$ if and only if $A \subseteq B$.

(61) For all subsets A, B of T such that A is a closed domain and B is a closed domain holds $\operatorname{Int} A \subseteq \operatorname{Int} B$ if and only if $A \subseteq B$.

(62) For all subsets A, B of T such that A is an open domain and B is an open domain holds $\overline{A} \subseteq \overline{B}$ if and only if $A \subseteq B$.

(63) For all subsets A, B of T such that A is a closed domain holds if $A \subseteq B$, then $\overline{\operatorname{Int}(A \cap B)} = A$.

(64) For all subsets A, B of T such that B is an open domain holds if $A \subseteq B$, then $\operatorname{Int} \overline{A \cup B} = B$.

Let us consider T. A family of subsets of T is domains-family if:

(Def.2) for every subset A of T such that $A \in$ it holds A is a domain.

The following propositions are true:

(65) For every family F of subsets of T holds $F \subseteq$ the domains of T if and only if F is domains-family.

(66) For every family F of subsets of T such that F is domains-family holds $\bigcup F \subseteq \overline{\mathrm{Int} \bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int} \overline{\bigcup F}}$.

(67) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int} \overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int} \overline{\mathrm{Int} \bigcap F} = \mathrm{Int} \bigcap F$.

(68) For every family F of subsets of T such that F is domains-family holds $\bigcup F \cup \mathrm{Int} \overline{\bigcup F}$ is a domain.

(69) Let F be a family of subsets of T. Then for every subset B of T such that $B \in F$ holds $B \subseteq \bigcup F \cup \mathrm{Int} \overline{\bigcup F}$ and for every subset A of T such that A is a domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\bigcup F \cup \mathrm{Int} \overline{\bigcup F} \subseteq A$.

(70) For every family F of subsets of T such that F is domains-family holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F}$ is a domain.

(71) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F} \subseteq B$,

(ii) $F = \emptyset$ or for every subset A of T such that A is a domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$.

Let us consider T. A family of subsets of T is closed-domains-family if:

(Def.3) for every subset A of T such that $A \in$ it holds A is a closed domain.

We now state several propositions:

(72) For every family F of subsets of T holds $F \subseteq$ the closed domains of T if and only if F is closed-domains-family.

(73) For every family F of subsets of T such that F is closed-domains-family holds F is domains-family.

(74) For every family F of subsets of T such that F is closed-domains-family holds F is closed.

(75) For every family F of subsets of T such that F is domains-family holds $\overline{F}$ is closed-domains-family.

(76) For every family F of subsets of T such that F is closed-domains-family holds $\overline{\bigcup F}$ is a closed domain and $\overline{\mathrm{Int} \bigcap F}$ is a closed domain.

(77) For every family F of subsets of T holds for every subset B of T such that $B \in F$ holds $B \subseteq \overline{\bigcup F}$ and for every subset A of T such that A is a closed domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\overline{\bigcup F} \subseteq A$.

(78) Let F be a family of subsets of T. Then if F is closed, then for every subset B of T such that $B \in F$ holds $\overline{\mathrm{Int} \bigcap F} \subseteq B$ but $F = \emptyset$ or for every subset A of T such that A is a closed domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \overline{\mathrm{Int} \bigcap F}$.

Let us consider T. A family of subsets of T is open-domains-family if:

(Def.4) for every subset A of T such that $A \in$ it holds A is an open domain.

We now state several propositions:

(79) For every family F of subsets of T holds $F \subseteq$ the open domains of T if and only if F is open-domains-family.

(80) For every family F of subsets of T such that F is open-domains-family holds F is domains-family.

(81) For every family F of subsets of T such that F is open-domains-family holds F is open.

(82) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int}\, \overline{F}$ is open-domains-family.

(83) For every family F of subsets of T such that F is open-domains-family holds $\mathrm{Int}\, \overline{\bigcap F}$ is an open domain and $\mathrm{Int}\, \overline{\bigcup F}$ is an open domain.

(84) For every family F of subsets of T holds if F is open, then for every subset B of T such that $B \in F$ holds $B \subseteq \mathrm{Int}\, \overline{\bigcup F}$ but for every subset A of T such that A is an open domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\mathrm{Int}\, \overline{\bigcup F} \subseteq A$.

(85) For every family F of subsets of T holds for every subset B of T such that $B \in F$ holds $\mathrm{Int}\, \overline{\bigcap F} \subseteq B$ but $F = \emptyset$ or for every subset A of T such that A is an open domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \mathrm{Int}\, \overline{\bigcap F}$.

4. Completeness of the Lattice of Domains

In the sequel T denotes a topological space. Next we state several propositions:

(86) The carrier of the lattice of domains of $T =$ the domains of T.

(87) For all elements a, b of the lattice of domains of T and for all elements A, B of the domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = \mathrm{Int}\, \overline{A \cup B} \cup (A \cup B)$ and $a \sqcap b = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

(88) $\perp_{\text{the lattice of domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of domains of } T} = \Omega_T$.

(89) For all elements a, b of the lattice of domains of T and for all elements A, B of the domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(90) For every subset X of the lattice of domains of T there exists an element a of the lattice of domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(91) The lattice of domains of T is complete.

(92) For every family F of subsets of T such that F is domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of domains of } T)} X = \bigcup F \cup \mathrm{Int}\, \overline{\bigcup F}$.

(93) For every family F of subsets of T such that F is domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds

if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \bigcap F \cap \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

5. Completeness of the Lattices of Closed Domains and Open Domains

In the sequel T will be a topological space. The following propositions are true:

(94) The carrier of the lattice of closed domains of T = the closed domains of T.

(95) For all elements a, b of the lattice of closed domains of T and for all elements A, B of the closed domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = A \cup B$ and $a \sqcap b = \overline{\operatorname{Int}(A \cap B)}$.

(96) $\perp_{\text{the lattice of closed domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of closed domains of } T} = \Omega_T$.

(97) For all elements a, b of the lattice of closed domains of T and for all elements A, B of the closed domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(98) For every subset X of the lattice of closed domains of T there exists an element a of the lattice of closed domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of closed domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(99) The lattice of closed domains of T is complete.

(100) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of closed domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of closed domains of } T)} X = \overline{\bigcup F}$.

(101) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of closed domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \Omega_T$.

(102) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

(103) The carrier of the lattice of open domains of T = the open domains of T.

(104) For all elements a, b of the lattice of open domains of T and for all elements A, B of the open domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = \operatorname{Int} \overline{A \cup B}$ and $a \sqcap b = A \cap B$.

(105) $\perp_{\text{the lattice of open domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of open domains of } T} = \Omega_T$.

(106) For all elements a, b of the lattice of open domains of T and for all elements A, B of the open domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(107) For every subset X of the lattice of open domains of T there exists an element a of the lattice of open domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of open domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(108) The lattice of open domains of T is complete.

(109) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of open domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of open domains of } T)} X = \operatorname{Int} \overline{\bigcup F}$.

(110) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of open domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \operatorname{Int} \bigcap F$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \Omega_T$.

(111) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of domains of } T)} X = \operatorname{Int} \overline{\bigcup F}$.

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References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Garrett Birkhoff. *Lattice Theory*. Providence, Rhode Island, New York, 1967.
[6] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Formalized Mathematics*, 2(**4**):453–459, 1991.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Ryszard Engelking. *General Topology*. Volume 60 of *Monografie Matematyczne*, PWN - Polish Scientific Publishers, Warsaw, 1977.
[14] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.

[15] Kazimierz Kuratowski. Sur l'opération $\overline{A}$ de l'analysis situs. *Fundamenta Mathematicae*, 3:182–199, 1922.
[16] Kazimierz Kuratowski. *Topology*. Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[17] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory (with an introduction to descriptive set theory)*. Volume 86 of *Studies in Logic and The Foundations of Mathematics*, PWN - Polish Scientific Publishers and North-Holland Publishing Company, Warsaw-Amsterdam, 1976.
[18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[23] Toshihiko Watanabe. The lattice of domains of a topological space. *Formalized Mathematics*, 3(**1**):41–46, 1992.
[24] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.
[25] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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