FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Connectedness Conditions Using Polygonal Arcs

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Summary. A concept of special polygonal arc joining two different points is defined. Any two points in a ball can be connected by this kind of arc, and that is also true for any region in $\mathcal{E}^2_{\rm T}$.

MML Identifier: `TOPREAL4`.

The notation and terminology used here have been introduced in the following articles: [13], [9], [1], [4], [2], [12], [11], [14], [10], [5], [3], [6], [7], and [8]. For simplicity we follow a convention: P, P_1, P_2, R will denote subsets of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q will denote points of $\mathcal{E}^2_{\rm T}$, f, h will denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r will denote a real number, u will denote a point of $\mathcal{E}^2$, and n, i will denote natural numbers. We now define three new predicates. Let us consider P, p, q. We say that P is a special polygonal arc joining p and q if and only if:

(Def.1) there exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and $p = f(1)$ and $q = f(\mathrm{len}\, f)$.

Let us consider P. We say that P is a special polygon if and only if the conditions (Def.2) is satisfied.

(Def.2) (i) There exist p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$,

(ii) for all p, q such that $p \in P$ and $q \in P$ and $p \neq q$ there exist P_1, P_2 such that P_1 is a special polygonal arc joining p and q and P_2 is a special polygonal arc joining p and q and $P_1 \cap P_2 = \{p, q\}$ and $P = P_1 \cup P_2$.

We say that P is a region if and only if:

(Def.3) P is open and P is connected.

The following propositions are true:

[1]The article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

(1) If P is a special polygonal arc joining p and q, then P is a special polygonal arc.

(2) If P is a special polygonal arc joining p and q, then P is an arc from p to q.

(3) If P is a special polygonal arc joining p and q, then $p \in P$ and $q \in P$.

(4) If P is a special polygonal arc joining p and q, then $p \neq q$.

(5) If P is a special polygon, then P is a simple closed curve.

(6) Suppose $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(7) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(8) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(9) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(10) If $r > 0$ and $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq \mathrm{Ball}(u,r)$.

(11) Suppose $p \neq p_1$ and $p_{1\mathbf{2}} = p_{\mathbf{2}}$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \in \mathcal{L}(f,1,2)$ and $h = \langle p_1, [\frac{p_{1\mathbf{1}}+p_{\mathbf{1}}}{2}, p_{1\mathbf{2}}], p\rangle$. Then h is a special sequence and $h(1) = p_1$ and $h(\mathrm{len}\, h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$.

(12) Suppose $p \neq p_1$ and $p_{1\mathbf{1}} = p_{\mathbf{1}}$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \in \mathcal{L}(f,1,2)$ and $h = \langle p_1, [p_{1\mathbf{1}}, \frac{p_{1\mathbf{2}}+p_{\mathbf{2}}}{2}], p\rangle$. Then h is a special sequence and $h(1) = p_1$ and $h(\mathrm{len}\, h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$.

(13) Suppose that

(i) $p \neq p_1$,

(ii) f is a special sequence,

(iii) $f(1) = p_1$,

(iv) $f(\mathrm{len}\, f) = p_2$,

(v) $i \in \mathrm{dom}\, f$,

(vi) $i+1 \in \operatorname{dom} f$,
(vii) $i > 1$,
(viii) $p \in \mathcal{L}(f, i, i+1)$,
(ix) $p \neq f(i)$,
(x) $p \neq f(i+1)$,
(xi) $h = (f \restriction i) \frown \langle p \rangle$,
(xii) $q = f(i)$.

Then h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(q, p)$.

(14) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(2) = p$ and $p_{\mathbf{2}} = p_{1\mathbf{2}}$ and $h = \langle p_1, [\frac{p_{1\mathbf{1}} + p_{\mathbf{1}}}{2}, p_{1\mathbf{2}}], p \rangle$. Then

(i) h is a special sequence,
(ii) $h(1) = p_1$,
(iii) $h(\operatorname{len} h) = p$,
(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p,
(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$,
(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$,
(vii) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 2) \cup \mathcal{L}(p, p)$.

(15) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(2) = p$ and $p_{\mathbf{1}} = p_{1\mathbf{1}}$ and $h = \langle p_1, [p_{1\mathbf{1}}, \frac{p_{1\mathbf{2}} + p_{\mathbf{2}}}{2}], p \rangle$. Then

(i) h is a special sequence,
(ii) $h(1) = p_1$,
(iii) $h(\operatorname{len} h) = p$,
(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p,
(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$,
(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$,
(vii) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 2) \cup \mathcal{L}(p, p)$.

(16) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(i) = p$ and $i > 2$ and $i \in \operatorname{dom} f$ and $h = f \restriction i$. Then h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(p, p)$.

(17) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $p \in \mathcal{L}(f, n, n+1)$ and $q = f(n)$. Then there exists h such that h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction n) \cup \mathcal{L}(q, p)$.

(18) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $p \in \widetilde{\mathcal{L}}(f)$. Then there exists h such that h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$.

(19) Suppose that
(i) $p_{\mathbf{1}} = p2_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$ or $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$ and $p_{\mathbf{2}} = p2_{\mathbf{2}}$,
(ii) $r > 0$,
(iii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iv) $p_2 \in \mathrm{Ball}(u, r)$,
(v) $p \in \mathrm{Ball}(u, r)$,
(vi) f is a special sequence,
(vii) $f(1) = p_1$,
(viii) $f(\mathrm{len}\, f) = p_2$,
(ix) $\mathcal{L}(p_2, p) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$,
(x) $h = f \frown \langle p \rangle$.

Then h is a special sequence and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(20) Suppose that
(i) $r > 0$,
(ii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iii) $p_2 \in \mathrm{Ball}(u, r)$,
(iv) $p \in \mathrm{Ball}(u, r)$,
(v) $[p_{\mathbf{1}}, p2_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
(vi) f is a special sequence,
(vii) $f(1) = p_1$,
(viii) $f(\mathrm{len}\, f) = p_2$,
(ix) $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$,
(x) $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$,
(xi) $h = f \frown \langle [p_{\mathbf{1}}, p2_{\mathbf{2}}], p \rangle$,
(xii) $(\mathcal{L}(p_2, [p_{\mathbf{1}}, p2_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, p2_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$.

Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(21) Suppose that
(i) $r > 0$,
(ii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iii) $p_2 \in \mathrm{Ball}(u, r)$,
(iv) $p \in \mathrm{Ball}(u, r)$,
(v) $[p2_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
(vi) f is a special sequence,
(vii) $f(1) = p_1$,
(viii) $f(\mathrm{len}\, f) = p_2$,
(ix) $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$,
(x) $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$,
(xi) $h = f \frown \langle [p2_{\mathbf{1}}, p_{\mathbf{2}}], p \rangle$,
(xii) $(\mathcal{L}(p_2, [p2_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([p2_{\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$.

Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(22) Suppose $r > 0$ and $p_1 \notin \mathrm{Ball}(u, r)$ and $p_2 \in \mathrm{Ball}(u, r)$ and $p \in \mathrm{Ball}(u, r)$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \notin \widetilde{\mathcal{L}}(f)$. Then there exists h such that $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(23) Given R, p, p_1, p_2, P, r, u. Then if $p \neq p_1$ and P is a special polygonal arc joining p_1 and p_2 and $P \subseteq R$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $p_2 \in \mathrm{Ball}(u, r)$ and $\mathrm{Ball}(u, r) \subseteq R$, then there exists P_1 such that P_1 is a special polygonal arc joining p_1 and p and $P_1 \subseteq R$.

(24) For every p such that R is a region and $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$ holds P is open.

(25) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then P is open.

(26) If $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $P \subseteq R$.

(27) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $R \subseteq P$.

(28) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $R = P$.

(29) If R is a region and $p \in R$ and $q \in R$ and $p \neq q$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq R$.

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Received August 24, 1992