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The Brouwer Fixed Point Theorem for Intervals [1]

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Summary. The aim is to prove, using Mizar System, the following simplest version of the Brouwer Fixed Point Theorem [2]. *For every continuous mapping $f : \mathbb{I} \to \mathbb{I}$ of the topological unit interval $\mathbb{I}$ there exists a point x such that $f(x) = x$* (see e.g. [9], [3]).

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The terminology and notation used here are introduced in the following papers: [23], [22], [25], [16], [5], [6], [20], [4], [18], [10], [24], [14], [19], [17], [7], [15], [11], [1], [21], [8], [13], and [12].

1. Properties of Topological Intervals

The following three propositions are true:

(1) For all real numbers a, b, c, d such that $a \leq c$ and $d \leq b$ and $c \leq d$ holds $[c, d] \subseteq [a, b]$.

(2) For all real numbers a, b, c, d such that $a \leq c$ and $b \leq d$ and $c \leq b$ holds $[a, b] \cup [c, d] = [a, d]$.

(3) For all real numbers a, b, c, d such that $a \leq c$ and $b \leq d$ and $c \leq b$ holds $[a, b] \cap [c, d] = [c, b]$.

In the sequel a, b, c, d are real numbers. We now state four propositions:

(4) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = [a, b]$ holds A is closed.

(5) If $a \leq b$, then $[a,\, b]_{\mathrm{T}}$ is a closed subspace of $\mathbb{R}^{\mathbf{1}}$.

(6) If $a \leq c$ and $d \leq b$ and $c \leq d$, then $[c,\, d]_{\mathrm{T}}$ is a closed subspace of $[a,\, b]_{\mathrm{T}}$.

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(7) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a, d]_{\rm T} = [a, b]_{\rm T} \cup [c, d]_{\rm T}$ and $[c, b]_{\rm T} = [a, b]_{\rm T} \cap [c, d]_{\rm T}$.

We now define two new functors. Let a, b be real numbers. Let us assume that $a \leq b$. The functor $a_{[a,b]_{\rm T}}$ yields a point of $[a, b]_{\rm T}$ and is defined by:

(Def.1) $a_{[a,b]_{\rm T}} = a$.

The functor $b_{[a,b]_{\rm T}}$ yields a point of $[a, b]_{\rm T}$ and is defined by:

(Def.2) $b_{[a,b]_{\rm T}} = b$.

One can prove the following two propositions:

(8) $0_{\mathbb{I}} = 0_{[0,1]_{\rm T}}$ and $1_{\mathbb{I}} = 1_{[0,1]_{\rm T}}$.

(9) If $a \leq b$ and $b \leq c$, then $a_{[a,b]_{\rm T}} = a_{[a,c]_{\rm T}}$ and $c_{[b,c]_{\rm T}} = c_{[a,c]_{\rm T}}$.

2. Continuous Mappings Between Topological Intervals

Let a, b be real numbers satisfying the condition: $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\rm T}$. The functor ${\rm L}_{01}(t_1, t_2)$ yielding a mapping from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$ is defined as follows:

(Def.3) for every point s of $[0, 1]_{\rm T}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $({\rm L}_{01}(t_1, t_2))(s) = (1 - r) \cdot r_1 + r \cdot r_2$.

We now state four propositions:

(10) Let a, b be real numbers. Then if $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\rm T}$ and for every point s of $[0, 1]_{\rm T}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $({\rm L}_{01}(t_1, t_2))(s) = (r_2 - r_1) \cdot r + r_1$.

(11) For all real numbers a, b such that $a \leq b$ and for all points t_1, t_2 of $[a, b]_{\rm T}$ holds ${\rm L}_{01}(t_1, t_2)$ is a continuous mapping from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$.

(12) For all real numbers a, b such that $a \leq b$ and for all points t_1, t_2 of $[a, b]_{\rm T}$ holds $({\rm L}_{01}(t_1, t_2))(0_{[0,1]_{\rm T}}) = t_1$ and $({\rm L}_{01}(t_1, t_2))(1_{[0,1]_{\rm T}}) = t_2$.

(13) ${\rm L}_{01}(0_{[0,1]_{\rm T}}, 1_{[0,1]_{\rm T}}) = {\rm id}_{([0,\,1]_{\rm T})}$.

Let a, b be real numbers satisfying the condition: $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$. The functor ${\rm P}_{01}(a, b, t_1, t_2)$ yielding a mapping from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$ is defined as follows:

(Def.4) for every point s of $[a, b]_{\rm T}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $({\rm P}_{01}(a, b, t_1, t_2))(s) = \frac{(b-r) \cdot r_1 + (r-a) \cdot r_2}{b-a}$.

The following propositions are true:

(14) Let a, b be real numbers. Suppose $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$. Let s be a point of $[a, b]_{\rm T}$. Then for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $({\rm P}_{01}(a, b, t_1, t_2))(s) = \frac{r_2 - r_1}{b-a} \cdot r + \frac{b \cdot r_1 - a \cdot r_2}{b-a}$.

(15) For all real numbers a, b such that $a < b$ and for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds ${\rm P}_{01}(a, b, t_1, t_2)$ is a continuous mapping from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$.

(16) For all real numbers a, b such that $a < b$ and for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds $({\rm P}_{01}(a, b, t_1, t_2))(a_{[a,b]_{\rm T}}) = t_1$ and $({\rm P}_{01}(a, b, t_1, t_2))(b_{[a,b]_{\rm T}}) = t_2$.

(17) $\mathrm{P}_{01}(0, 1, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}) = \mathrm{id}_{([0,1]_{\mathrm{T}})}$.

(18) Let a, b be real numbers. Then if $a < b$, then $\mathrm{id}_{([a,b]_{\mathrm{T}})} = \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$ and $\mathrm{id}_{([0,1]_{\mathrm{T}})} = \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}) \cdot \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$.

(19) Let a, b be real numbers. Then if $a < b$, then $\mathrm{id}_{([a,b]_{\mathrm{T}})} = \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$ and $\mathrm{id}_{([0,1]_{\mathrm{T}})} = \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}) \cdot \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$.

(20) Let a, b be real numbers. Suppose $a < b$. Then
- (i) $\mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$ is a homeomorphism,
- (ii) $(\mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}}))^{-1} = \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$,
- (iii) $\mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$ is a homeomorphism,
- (iv) $(\mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}))^{-1} = \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$.

(21) Let a, b be real numbers. Suppose $a < b$. Then
- (i) $\mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$ is a homeomorphism,
- (ii) $(\mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}}))^{-1} = \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$,
- (iii) $\mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$ is a homeomorphism,
- (iv) $(\mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))^{-1} = \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$.

3. Connectedness of Intervals and Brouwer Fixed Point Theorem for Intervals

We now state several propositions:

(22) $\mathbb{I}$ is connected.

(23) For all real numbers a, b such that $a \leq b$ holds $[a, b]_{\mathrm{T}}$ is connected.

(24) For every continuous mapping f from $\mathbb{I}$ into $\mathbb{I}$ there exists a point x of $\mathbb{I}$ such that $f(x) = x$.

(25) For all real numbers a, b such that $a \leq b$ and for every continuous mapping f from $[a, b]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$ there exists a point x of $[a, b]_{\mathrm{T}}$ such that $f(x) = x$.

(26) Let X, Y be subspaces of $\mathbb{R}^{\mathbf{1}}$. Then for every continuous mapping f from X into Y such that there exist real numbers a, b such that $a \leq b$ and $[a, b] \subseteq$ the carrier of X and $[a, b] \subseteq$ the carrier of Y and $f\,^\circ\,[a, b] \subseteq [a, b]$ there exists a point x of X such that $f(x) = x$.

(27) For all subspaces X, Y of $\mathbb{R}^{\mathbf{1}}$ and for every continuous mapping f from X into Y such that there exist real numbers a, b such that $a \leq b$ and $[a, b] \subseteq$ the carrier of X and $f\,^\circ\,[a, b] \subseteq [a, b]$ there exists a point x of X such that $f(x) = x$.

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