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Sum and Product of Finite Sequences of Elements of a Field

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Summary. This article is concerned with a generalization of concepts introduced in [10], i.e., there are introduced the sum and the product of finite number of elements of any field. Moreover, the product of vectors which yields a vector is introduced. According to [10], some operations on i-tuples of elements of field are introduced: addition, subtraction, and complement. Some properties on the sum and the product of finite number of elements of a field are present.

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The articles [17], [2], [18], [7], [8], [3], [4], [14], [13], [15], [19], [16], [6], [5], [9], [1], [20], [22], [21], [11], and [12] provide the notation and terminology for this paper.

1. Auxiliary theorems

For simplicity we adopt the following convention: i, j, k will denote natural numbers, K will denote a field, a, a', a_1, a_2, a_3 will denote elements of the carrier of K, p, p_1, p_2, q will denote finite sequences of elements of the carrier of K, and R, R_1, R_2, R_3 will denote elements of (the carrier of K)^i. We now state a number of propositions:

(1) $-0_K = 0_K$.

(2) The addition of K is commutative.

(3) The addition of K is associative.

(4) The multiplication of K is commutative.

(5) The multiplication of K is associative.

(6) 1_K is a unity w.r.t. the multiplication of K.

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(7) $\mathbf{1}_{\text{the multiplication of } K} = 1_K$.

(8) 0_K is a unity w.r.t. the addition of K.

(9) $\mathbf{1}_{\text{the addition of } K} = 0_K$.

(10) The addition of K has a unity.

(11) The multiplication of K has a unity.

(12) The multiplication of K is distributive w.r.t. the addition of K.

We now define two new functors. Let us consider K, and let a be an element of the carrier of K. The functor $\cdot^a$ yields a unary operation on the carrier of K and is defined by:

(Def.1) $\cdot^a = (\text{the multiplication of } K)^\circ(a, \mathrm{id}_{(\text{the carrier of } K)})$.

Let us consider K. The functor $-_K$ yields a binary operation on the carrier of K and is defined as follows:

(Def.2) $-_K = (\text{the addition of } K) \circ (\mathrm{id}_{(\text{the carrier of } K)}, \text{the reverse-map of } K)$.

We now state several propositions:

(13) $-_K = (\text{the addition of } K) \circ (\mathrm{id}_{(\text{the carrier of } K)}, \text{the reverse-map of } K)$.

(14) $-_K(a_1, a_2) = a_1 - a_2$.

(15) $\cdot^a$ is distributive w.r.t. the addition of K.

(16) The reverse-map of K is an inverse operation w.r.t. the addition of K.

(17) The addition of K has an inverse operation.

(18) The inverse operation w.r.t. the addition of K = the reverse-map of K.

(19) The reverse-map of K is distributive w.r.t. the addition of K.

Let us consider K, p_1, p_2. The functor $p_1 + p_2$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.3) $p_1 + p_2 = (\text{the addition of } K)^\circ(p_1, p_2)$.

Next we state two propositions:

(20) $p_1 + p_2 = (\text{the addition of } K)^\circ(p_1, p_2)$.

(21) If $i \in \mathrm{Seg}\,\mathrm{len}(p_1 + p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 + p_2)(i) = a_1 + a_2$.

Let us consider i, and let us consider K, and let R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 + R_2$ is an element of (the carrier of $K)^i$.

Next we state several propositions:

(22) If $j \in \mathrm{Seg}\,i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 + R_2)(j) = a_1 + a_2$.

(23) $\varepsilon_{(\text{the carrier of } K)} + p = \varepsilon_{(\text{the carrier of } K)}$ and $p + \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(24) $\langle a_1 \rangle + \langle a_2 \rangle = \langle a_1 + a_2 \rangle$.

(25) $(i \longmapsto a_1) + (i \longmapsto a_2) = i \longmapsto a_1 + a_2$.

(26) $R_1 + R_2 = R_2 + R_1$.

(27) $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.

(28) $R + (i \longmapsto 0_K) = R$ and $R = (i \longmapsto 0_K) + R$.

Let us consider K, p. The functor $-p$ yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.4) $-p = (\text{the reverse-map of } K) \cdot p$.

The following two propositions are true:

(29) $-p = (\text{the reverse-map of } K) \cdot p$.

(30) If $i \in \mathrm{Seg}\,\mathrm{len}(-p)$ and $a = p(i)$, then $(-p)(i) = -a$.

Let us consider i, K, R. Then $-R$ is an element of (the carrier of $K)^i$.

One can prove the following propositions:

(31) If $j \in \mathrm{Seg}\,i$ and $a = R(j)$, then $(-R)(j) = -a$.

(32) $-\varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(33) $-\langle a \rangle = \langle -a \rangle$.

(34) $-(i \longmapsto a) = i \longmapsto -a$.

(35) $R + -R = i \longmapsto 0_K$ and $-R + R = i \longmapsto 0_K$.

(36) If $R_1 + R_2 = i \longmapsto 0_K$, then $R_1 = -R_2$ and $R_2 = -R_1$.

(37) $--R = R$.

(38) If $-R_1 = -R_2$, then $R_1 = R_2$.

(39) If $R_1 + R = R_2 + R$ or $R_1 + R = R + R_2$, then $R_1 = R_2$.

(40) $-(R_1 + R_2) = -R_1 + -R_2$.

Let us consider K, p_1, p_2. The functor $p_1 - p_2$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.5) $p_1 - p_2 = (-_K)^\circ(p_1, p_2)$.

Next we state two propositions:

(41) $p_1 - p_2 = (-_K)^\circ(p_1, p_2)$.

(42) If $i \in \mathrm{Seg}\,\mathrm{len}(p_1 - p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 - p_2)(i) = a_1 - a_2$.

Let us consider i, K, R_1, R_2. Then $R_1 - R_2$ is an element of (the carrier of $K)^i$.

The following propositions are true:

(43) If $j \in \mathrm{Seg}\,i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 - R_2)(j) = a_1 - a_2$.

(44) $\varepsilon_{(\text{the carrier of } K)} - p = \varepsilon_{(\text{the carrier of } K)}$ and $p - \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(45) $\langle a_1 \rangle - \langle a_2 \rangle = \langle a_1 - a_2 \rangle$.

(46) $(i \longmapsto a_1) - (i \longmapsto a_2) = i \longmapsto a_1 - a_2$.

(47) $R_1 - R_2 = R_1 + -R_2$.

(48) $R - (i \longmapsto 0_K) = R$.

(49) $(i \longmapsto 0_K) - R = -R$.

(50) $R_1 - -R_2 = R_1 + R_2$.

(51) $-(R_1 - R_2) = R_2 - R_1$.

(52) $-(R_1 - R_2) = -R_1 + R_2$.

(53) $R - R = i \longmapsto 0_K$.

(54) If $R_1 - R_2 = i \longmapsto 0_K$, then $R_1 = R_2$.

(55) $R_1 - R_2 - R_3 = R_1 - (R_2 + R_3)$.

(56) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(57) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(58) $R_1 = (R_1 + R) - R$.

(59) $R_1 = (R_1 - R) + R$.

(60) For all elements a, b of the carrier of K holds ((the multiplication of $K)^\circ(a, \mathrm{id}_{(\text{the carrier of } K)}))(b) = a \cdot b$.

(61) For all elements a, b of the carrier of K holds $\cdot^a(b) = a \cdot b$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K, and let a be an element of the carrier of K. The functor $a \cdot p$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.6) $a \cdot p = \cdot^a \cdot p$.

Next we state the proposition

(62) If $i \in \mathrm{Seg}\,\mathrm{len}(a \cdot p)$ and $a' = p(i)$, then $(a \cdot p)(i) = a \cdot a'$.

Let us consider i, K, R, a. Then $a \cdot R$ is an element of (the carrier of $K)^i$.

The following propositions are true:

(63) If $j \in \mathrm{Seg}\,i$ and $a' = R(j)$, then $(a \cdot R)(j) = a \cdot a'$.

(64) $a \cdot \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(65) $a \cdot \langle a_1 \rangle = \langle a \cdot a_1 \rangle$.

(66) $a_1 \cdot (i \longmapsto a_2) = i \longmapsto a_1 \cdot a_2$.

(67) $(a_1 \cdot a_2) \cdot R = a_1 \cdot (a_2 \cdot R)$.

(68) $(a_1 + a_2) \cdot R = a_1 \cdot R + a_2 \cdot R$.

(69) $a \cdot (R_1 + R_2) = a \cdot R_1 + a \cdot R_2$.

(70) $1_K \cdot R = R$.

(71) $0_K \cdot R = i \longmapsto 0_K$.

(72) $(-1_K) \cdot R = -R$.

Let us consider K, p_1, p_2. The functor $p_1 \bullet p_2$ yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.7) $p_1 \bullet p_2 =$ (the multiplication of $K)^\circ(p_1, p_2)$.

One can prove the following proposition

(73) If $i \in \mathrm{Seg}\,\mathrm{len}(p_1 \bullet p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 \bullet p_2)(i) = a_1 \cdot a_2$.

Let us consider i, K, R_1, R_2. Then $R_1 \bullet R_2$ is an element of (the carrier of $K)^i$.

We now state a number of propositions:

(74) If $j \in \mathrm{Seg}\,i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 \bullet R_2)(j) = a_1 \cdot a_2$.

(75) $\varepsilon_{(\text{the carrier of } K)} \bullet p = \varepsilon_{(\text{the carrier of } K)}$ and $p \bullet \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(76) $\langle a_1 \rangle \bullet \langle a_2 \rangle = \langle a_1 \cdot a_2 \rangle$.

(77) $R_1 \bullet R_2 = R_2 \bullet R_1$.

(78) $p \bullet q = q \bullet p$.

(79) $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

(80) $(i \longmapsto a) \bullet R = a \cdot R$ and $R \bullet (i \longmapsto a) = a \cdot R$.

(81) $(i \longmapsto a_1) \bullet (i \longmapsto a_2) = i \longmapsto a_1 \cdot a_2$.

(82) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$.

(83) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$ and $a \cdot (R_1 \bullet R_2) = R_1 \bullet a \cdot R_2$.

(84) $a \cdot R = (i \longmapsto a) \bullet R$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K. The functor $\sum p$ yielding an element of the carrier of K is defined as follows:

(Def.8) $\sum p =$ the addition of $K \circledast p$.

The following propositions are true:

(85) $\sum(\varepsilon_{(\text{the carrier of } K)}) = 0_K$.

(86) $\sum\langle a \rangle = a$.

(87) $\sum(p \frown \langle a \rangle) = \sum p + a$.

(88) $\sum(p_1 \frown p_2) = \sum p_1 + \sum p_2$.

(89) $\sum(\langle a \rangle \frown p) = a + \sum p$.

(90) $\sum\langle a_1, a_2 \rangle = a_1 + a_2$.

(91) $\sum\langle a_1, a_2, a_3 \rangle = a_1 + a_2 + a_3$.

(92) $\sum(a \cdot p) = a \cdot \sum p$.

(93) For every element R of (the carrier of $K)^0$ holds $\sum R = 0_K$.

(94) $\sum(-p) = -\sum p$.

(95) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(96) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K. The functor $\prod p$ yielding an element of the carrier of K is defined by:

(Def.9) $\prod p =$ the multiplication of $K \circledast p$.

The following propositions are true:

(97) $\prod p =$ the multiplication of $K \circledast p$.

(98) $\prod(\varepsilon_{(\text{the carrier of } K)}) = 1_K$.

(99) $\prod\langle a \rangle = a$.

(100) $\prod(p \frown \langle a \rangle) = \prod p \cdot a$.

(101) $\prod(p_1 \frown p_2) = \prod p_1 \cdot \prod p_2$.

(102) $\prod(\langle a \rangle \frown p) = a \cdot \prod p$.

(103) $\prod\langle a_1, a_2 \rangle = a_1 \cdot a_2$.

(104) $\prod\langle a_1, a_2, a_3 \rangle = a_1 \cdot a_2 \cdot a_3$.

(105) For every element R of (the carrier of $K)^0$ holds $\prod R = 1_K$.

(106) $\prod(i \longmapsto 1_K) = 1_K$.

(107) There exists k such that $k \in \mathrm{Seg}\,\mathrm{len}\,p$ and $p(k) = 0_K$ if and only if $\prod p = 0_K$.

(108) $\prod(i + j \longmapsto a) = \prod(i \longmapsto a) \cdot \prod(j \longmapsto a)$.

(109) $\prod(i \cdot j \longmapsto a) = \prod(j \longmapsto \prod(i \longmapsto a))$.

(110) $\prod(i \longmapsto a_1 \cdot a_2) = \prod(i \longmapsto a_1) \cdot \prod(i \longmapsto a_2)$.

(111) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(112) $\prod(a \cdot R) = \prod(i \longmapsto a) \cdot \prod R$.

Let us consider K, and let p, q be finite sequences of elements of the carrier of K. The functor $p \cdot q$ yielding an element of the carrier of K is defined by:

(Def.10) $p \cdot q = \sum(p \bullet q)$.

One can prove the following propositions:

(113) For all elements a, b of the carrier of K holds $\langle a \rangle \cdot \langle b \rangle = a \cdot b$.

(114) For all elements a_1, a_2, b_1, b_2 of the carrier of K holds $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 \cdot b_1 + a_2 \cdot b_2$.

(115) For all finite sequences p, q of elements of the carrier of K holds $p \cdot q = q \cdot p$.

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