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Properties of the Intervals of Real Numbers

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Summary. The paper contains definitions and basic properties of the intervals of real numbers.

The article includes the text being a continuation of the paper [5]. Some theorems concerning basic properties of intervals are proved.

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The notation and terminology used here are introduced in the following papers: [16], [15], [11], [12], [9], [10], [1], [14], [2], [13], [4], [6], [8], [7], [3], [5], and [17]. The following propositions are true:

(1) For all *Real numbers* x, y such that $x \neq -\infty$ and $x \neq +\infty$ and $x \leq y$ holds $0_{\overline{\mathbb{R}}} \leq y - x$.

(2) For all *Real numbers* x, y such that it is not true that: $x = -\infty$ and $y = -\infty$ and it is not true that: $x = +\infty$ and $y = +\infty$ and $x \leq y$ holds $0_{\overline{\mathbb{R}}} \leq y - x$.

(3) For all *Real numbers* x, y holds $x \leq y$ or $y \leq x$.

(4) For all *Real numbers* x, y such that $x \neq y$ holds $x < y$ or $y < x$.

(5) For all *Real numbers* x, y holds $x < y$ or $y \leq x$.

(6) For all *Real numbers* x, y holds $x < y$ if and only if $y \not\leq x$.

(7) For all *Real numbers* x, y, z such that $x < y$ and $y < z$ holds $x < z$.

(8) For all *Real numbers* a, b, c such that $b \neq -\infty$ and $b \neq +\infty$ and it is not true that: $a = -\infty$ and $c = -\infty$ and it is not true that: $a = +\infty$ and $c = +\infty$ holds $(c - b) + (b - a) = c - a$.

(9) For all *Real numbers* a_1, a_2 holds $\inf\{a_1, a_2\} \leq a_1$ and $\inf\{a_1, a_2\} \leq a_2$ and $a_1 \leq \sup\{a_1, a_2\}$ and $a_2 \leq \sup\{a_1, a_2\}$.

(10) For all *Real numbers* a, b, c such that $a \leq b$ and $b < c$ or $a < b$ and $b \leq c$ holds $a < c$.

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We now define several new constructions. Let a, b be *Real numbers*. The functor $[a, b]$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def.1) for every *Real number* x holds $x \in [a, b]$ if and only if $a \leq x$ and $x \leq b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $]a, b[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.2) for every *Real number* x holds $x \in]a, b[$ if and only if $a < x$ and $x < b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $]a, b]$ yielding a subset of $\mathbb{R}$ is defined by:

(Def.3) for every *Real number* x holds $x \in]a, b]$ if and only if $a < x$ and $x \leq b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $[a, b[$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.4) for every *Real number* x holds $x \in [a, b[$ if and only if $a \leq x$ and $x < b$ and $x \in \mathbb{R}$.

A subset of $\mathbb{R}$ is called an open interval if:

(Def.5) there exist *Real numbers* a, b such that $a \leq b$ and it $=]a, b[$.

A subset of $\mathbb{R}$ is said to be a closed interval if:

(Def.6) there exist *Real numbers* a, b such that $a \leq b$ and it $= [a, b]$.

A subset of $\mathbb{R}$ is said to be a right-open interval if:

(Def.7) there exist *Real numbers* a, b such that $a \leq b$ and it $= [a, b[$.

A subset of $\mathbb{R}$ is called a left-open interval if:

(Def.8) there exist *Real numbers* a, b such that $a \leq b$ and it $=]a, b]$.

A subset of $\mathbb{R}$ is said to be an interval if:

(Def.9) it is an open interval or it is a closed interval or it is a right-open interval or it is a left-open interval.

We see that the open interval is an interval. We see that the closed interval is an interval. We see that the right-open interval is an interval. We see that the left-open interval is an interval.

We now state a number of propositions:

(11) For an arbitrary x and for all *Real numbers* a, b such that $x \in]a, b[$ or $x \in [a, b]$ or $x \in [a, b[$ or $x \in]a, b]$ holds x is a *Real number*.

(12) For all *Real numbers* a, b such that $b < a$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.

(13) For every *Real number* a holds $]a, a[= \emptyset$ and $[a, a[= \emptyset$ and $]a, a] = \emptyset$.

(14) For every *Real number* a holds if $a = -\infty$ or $a = +\infty$, then $[a, a] = \emptyset$ and also if $a \neq -\infty$ and $a \neq +\infty$, then $[a, a] = \{a\}$.

(15) For all *Real numbers* a, b such that $b \leq a$ holds $]a, b[= \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$ and $[a, b] \subseteq \{a\}$ and $[a, b] \subseteq \{b\}$.

(16) For all *Real numbers* a, b, c such that $a < b$ and $b < c$ holds $b \in \mathbb{R}$.

(17) For all *Real numbers* a, b such that $a < b$ there exists a *Real number* x such that $a < x$ and $x < b$ and $x \in \mathbb{R}$.

(18) For all *Real numbers* a, b, c such that $a < b$ and $a < c$ there exists a *Real number* x such that $a < x$ and $x < b$ and $x < c$ and $x \in \mathbb{R}$.

(19) For all *Real numbers* a, b, c such that $a < c$ and $b < c$ there exists a *Real number* x such that $a < x$ and $b < x$ and $x < c$ and $x \in \mathbb{R}$.

(20) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2[$.

(21) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2[$.

(22) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2[$.

(23) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2[$.

(24) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2]$.

(25) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2]$.

(26) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2[$.

(27) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2[$.

(28) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2[$.

(29) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2[$.

(30) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2]$.

(31) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2]$.

(32) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2[$.

(33) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2[$.

(34) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2]$.

(35) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2]$.

(36) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2[$.

(37) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2[$.

(38) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2]$.

(39) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2]$.

(40) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2]$.

(41) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2]$.

Next we state a number of propositions:

(42) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2]$.

(43) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2]$.

(44) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2[$.

(45) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2[$.

(46) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2]$.

(47) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2]$.

(48) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2[$.

(49) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2[$.

(50) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2]$.

(51) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2]$.

(52) Let A be an interval. Let a_1, a_2, b_1, b_2 be *Real numbers.* Suppose that
(i) $a_1 < b_1$ or $a_2 < b_2$,
(ii) $A =]a_1, b_1[$ or $A = [a_1, b_1]$ or $A = [a_1, b_1[$ or $A =]a_1, b_1]$ and also $A =]a_2, b_2[$ or $A = [a_2, b_2]$ or $A = [a_2, b_2[$ or $A =]a_2, b_2]$.
Then $a_1 = a_2$ and $b_1 = b_2$.

Let A be an interval. The functor $\mathrm{vol}(A)$ yielding a *Real number* is defined as follows:

(Def.10) there exist *Real numbers* a, b such that $A =]a, b[$ or $A = [a, b]$ or $A = [a, b[$ or $A =]a, b]$ and also if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

One can prove the following propositions:

(53) For every open interval A and for all *Real numbers* a, b such that $A =]a, b[$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(54) For every closed interval A and for all *Real numbers* a, b such that $A = [a, b]$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(55) For every right-open interval A and for all *Real numbers* a, b such that $A = [a, b[$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(56) For every left-open interval A and for all *Real numbers* a, b such that $A =]a, b]$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(57) Let A be an interval. Let a, b, c be *Real numbers.* Suppose that
(i) $a = -\infty$,

(ii) $b \in \mathbb{R}$,
(iii) $c = +\infty$,
(iv) $A =]a, b[$ or $A =]b, c[$ or $A = [a, b]$ or $A = [b, c]$ or $A = [a, b[$ or $A = [b, c[$ or $A =]a, b]$ or $A =]b, c]$.
Then $\mathrm{vol}(A) = +\infty$.

(58) For every interval A and for all *Real numbers* a, b such that $a = -\infty$ and $b = +\infty$ and also $A =]a, b[$ or $A = [a, b]$ or $A = [a, b[$ or $A =]a, b]$ holds $\mathrm{vol}(A) = +\infty$.

(59) For every interval A and for every *Real number* a such that $A =]a, a[$ or $A = [a, a]$ or $A = [a, a[$ or $A =]a, a]$ holds $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

Let us note that there exists an empty interval.

Let us note that it makes sense to consider the following constant. Then $\emptyset$ is an empty interval.

Next we state four propositions:

(60) $\mathrm{vol}(\emptyset) = 0_{\overline{\mathbb{R}}}$.

(61) For all intervals A, B and for all *Real numbers* a, b such that $A \subseteq B$ and $B = [a, b]$ and $b \leq a$ holds $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$ and $\mathrm{vol}(B) = 0_{\overline{\mathbb{R}}}$.

(62) For all intervals A, B such that $A \subseteq B$ holds $\mathrm{vol}(A) \leq \mathrm{vol}(B)$.

(63) For every interval A holds $0_{\overline{\mathbb{R}}} \leq \mathrm{vol}(A)$.

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