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Functions and Finite Sequences of Real Numbers

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Summary. We define notions of fiberwise equipotent functions, non-increasing finite sequences of real numbers and new operations on finite sequences. Equivalent conditions for fiberwise equivalent functions and basic facts about new constructions are shown.

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The articles [11], [4], [5], [3], [1], [8], [10], [2], [12], [6], [7], and [9] provide the notation and terminology for this paper. In the sequel n will be a natural number. Let F, G be functions. We say that F and G are fiwerwise equipotent if and only if:

(Def.1) for an arbitrary x holds $\overline{\overline{F^{-1}\{x\}}} = \overline{\overline{G^{-1}\{x\}}}$.

Let us observe that the predicate defined above is reflexive and symmetric.

One can prove the following propositions:

(1) For all functions F, G such that F and G are fiwerwise equipotent holds $\operatorname{rng} F = \operatorname{rng} G$.

(2) For all functions F, G, H such that F and G are fiwerwise equipotent and F and H are fiwerwise equipotent holds G and H are fiwerwise equipotent.

(3) For all functions F, G holds F and G are fiwerwise equipotent if and only if there exists a function H such that $\operatorname{dom} H = \operatorname{dom} F$ and $\operatorname{rng} H = \operatorname{dom} G$ and H is one-to-one and $F = G \cdot H$.

(4) For all functions F, G holds F and G are fiwerwise equipotent if and only if for every set X holds $\overline{\overline{F^{-1}X}} = \overline{\overline{G^{-1}X}}$.

(5) For every non-empty set D and for all functions F, G such that $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$ holds F and G are fiwerwise equipotent if and only if for every element d of D holds $\overline{\overline{F^{-1}\{d\}}} = \overline{\overline{G^{-1}\{d\}}}$.

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(6) For all functions F, G such that $\operatorname{dom} F = \operatorname{dom} G$ holds F and G are fiwerwise equipotent if and only if there exists a permutation P of $\operatorname{dom} F$ such that $F = G \cdot P$.

(7) For all functions F, G such that F and G are fiwerwise equipotent holds $\overline{\overline{\operatorname{dom} F}} = \overline{\overline{\operatorname{dom} G}}$.

(8) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for an arbitrary x holds $\operatorname{card}(F^{-1} \{x\}) = \operatorname{card}(G^{-1} \{x\})$.

(9) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for every set X holds $\operatorname{card}(F^{-1} X) = \operatorname{card}(G^{-1} X)$.

(10) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite and F and G are fiwerwise equipotent holds $\operatorname{card} \operatorname{dom} F = \operatorname{card} \operatorname{dom} G$.

(11) For every non-empty set D and for all functions F, G such that $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$ and $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for every element d of D holds $\operatorname{card}(F^{-1} \{d\}) = \operatorname{card}(G^{-1} \{d\})$.

(12) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if for an arbitrary x holds $\operatorname{card}(f^{-1} \{x\}) = \operatorname{card}(g^{-1} \{x\})$.

(13) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if for every set X holds $\operatorname{card}(f^{-1} X) = \operatorname{card}(g^{-1} X)$.

(14) For all finite sequences f, g, h holds f and g are fiwerwise equipotent if and only if $f \frown h$ and $g \frown h$ are fiwerwise equipotent.

(15) For all finite sequences f, g holds $f \frown g$ and $g \frown f$ are fiwerwise equipotent.

(16) For all finite sequences f, g such that f and g are fiwerwise equipotent holds $\operatorname{len} f = \operatorname{len} g$ and $\operatorname{dom} f = \operatorname{dom} g$.

(17) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if there exists a permutation P of $\operatorname{dom} g$ such that $f = g \cdot P$.

(18) For every function F and for every finite set X there exists a finite sequence f such that $F \restriction X$ and f are fiwerwise equipotent.

Let D be a non-empty set, and let f be a finite sequence of elements of D, and let n be a natural number. The functor $f_{\lfloor n}$ yields a finite sequence of elements of D and is defined as follows:

(Def.2) (i) $\operatorname{len}(f_{\lfloor n}) = \operatorname{len} f - n$ and for every natural number m such that $m \in \operatorname{dom}(f_{\lfloor n})$ holds $f_{\lfloor n}(m) = f(m + n)$ if $n \leq \operatorname{len} f$,

(ii) $f_{\lfloor n} = \varepsilon_D$, otherwise.

The following propositions are true:

(19) For every non-empty set D and for every finite sequence f of elements of D and for all natural numbers n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$ holds $(f \restriction n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

(20) For every non-empty set D and for every finite sequence f of elements of D and for every natural number n and for an arbitrary x such that

$\operatorname{len} f = n + 1$ and $x = f(n+1)$ holds $f = (f \restriction n) \frown \langle x \rangle$.

(21) For every non-empty set D and for every finite sequence f of elements of D and for every natural number n holds $(f \restriction n) \frown (f_{\lfloor n}) = f$.

(22) For all finite sequences R_1, R_2 of elements of $\mathbb{R}$ such that R_1 and R_2 are fiwerwise equipotent holds $\sum R_1 = \sum R_2$.

Let R be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{MIM}(R)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the conditions (Def.3).

(Def.3) (i) $\operatorname{len} \mathrm{MIM}(R) = \operatorname{len} R$,

(ii) $(\mathrm{MIM}(R))(\operatorname{len} \mathrm{MIM}(R)) = R(\operatorname{len} R)$,

(iii) for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{MIM}(R) - 1$ and for all real numbers r, s such that $R(n) = r$ and $R(n+1) = s$ holds $(\mathrm{MIM}(R))(n) = r - s$.

Next we state several propositions:

(23) For every finite sequence R of elements of $\mathbb{R}$ and for every real number r and for every natural number n such that $\operatorname{len} R = n+2$ and $R(n+1) = r$ holds $\mathrm{MIM}(R \restriction (n+1)) = (\mathrm{MIM}(R) \restriction n) \frown \langle r \rangle$.

(24) For every finite sequence R of elements of $\mathbb{R}$ and for all real numbers r, s and for every natural number n such that $\operatorname{len} R = n+2$ and $R(n+1) = r$ and $R(n+2) = s$ holds $\mathrm{MIM}(R) = (\mathrm{MIM}(R) \restriction n) \frown \langle r - s, s \rangle$.

(25) $\mathrm{MIM}(\varepsilon_{\mathbb{R}}) = \varepsilon_{\mathbb{R}}$.

(26) For every real number r holds $\mathrm{MIM}(\langle r \rangle) = \langle r \rangle$.

(27) For all real numbers r, s holds $\mathrm{MIM}(\langle r, s \rangle) = \langle r - s, s \rangle$.

(28) For every finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $(\mathrm{MIM}(R))_{\lfloor n} = \mathrm{MIM}(R_{\lfloor n})$.

(29) For every finite sequence R of elements of $\mathbb{R}$ such that $\operatorname{len} R \neq 0$ holds $\sum \mathrm{MIM}(R) = R(1)$.

(30) For every finite sequence R of elements of $\mathbb{R}$ and for every natural number n such that $1 \leq n$ and $n < \operatorname{len} R$ holds $\sum \mathrm{MIM}(R_{\lfloor n}) = R(n+1)$.

A finite sequence of elements of $\mathbb{R}$ is non-increasing if:

(Def.4) for every natural number n such that $n \in \operatorname{dom} \mathrm{it}$ and $n+1 \in \operatorname{dom} \mathrm{it}$ and for all real numbers r, s such that $r = \mathrm{it}(n)$ and $s = \mathrm{it}(n+1)$ holds $r \geq s$.

One can check that there exists a non-increasing finite sequence of elements of $\mathbb{R}$.

We now state several propositions:

(31) For every finite sequence R of elements of $\mathbb{R}$ such that $\operatorname{len} R = 0$ or $\operatorname{len} R = 1$ holds R is non-increasing.

(32) For every finite sequence R of elements of $\mathbb{R}$ holds R is non-increasing if and only if for all natural numbers n, m such that $n \in \operatorname{dom} R$ and $m \in \operatorname{dom} R$ and $n < m$ and for all real numbers r, s such that $R(n) = r$ and $R(m) = s$ holds $r \geq s$.

(33) For every non-increasing finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $R \restriction n$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(34) For every non-increasing finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $R_{\restriction n}$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(35) For every finite sequence R of elements of $\mathbb{R}$ there exists a non-increasing finite sequence R_1 of elements of $\mathbb{R}$ such that R and R_1 are fiwerwise equipotent.

(36) For all non-increasing finite sequences R_1, R_2 of elements of $\mathbb{R}$ such that R_1 and R_2 are fiwerwise equipotent holds $R_1 = R_2$.

(37) For every finite sequence R of elements of $\mathbb{R}$ and for all real numbers r, s such that $r \neq 0$ holds $R^{-1}\{\frac{s}{r}\} = (r \cdot R)^{-1}\{s\}$.

(38) For every finite sequence R of elements of $\mathbb{R}$ holds $(0 \cdot R)^{-1}\{0\} = \operatorname{dom} R$.

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