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Properties of Partial Functions from a Domain to the Set of Real Numbers

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Summary. The article consists of two parties. In the first one we consider notion of nonnegative and nonpositive part of a real numbers. In the second we consider partial function from a domain to the set of real numbers (or more general to a domain). We define a few new operations for these functions and show connections between finite sequences of real numbers and functions which domain is finite. We introduce *integrations* for finite domain real valued functions.

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The articles [23], [25], [7], [21], [3], [4], [1], [11], [13], [2], [18], [20], [22], [6], [24], [8], [5], [9], [10], [19], [16], [17], [15], [12], and [14] provide the notation and terminology for this paper.

1. Nonnegative and Nonpositive Part of a Real Number

In the sequel n is a natural number and r is a real number. We now define two new functors. Let n, m be natural numbers. Then $\min(n, m)$ is a natural number. Let r be a real number. The functor $\max_+(r)$ yielding a real number is defined as follows:

(Def.1) $\max_+(r) = \max(r, 0)$.

The functor $\max_-(r)$ yielding a real number is defined as follows:

(Def.2) $\max_-(r) = \max(-r, 0)$.

We now state several propositions:

(1) For every real number r holds $r = \max_+(r) - \max_-(r)$.

(2) For every real number r holds $|r| = \max_+(r) + \max_-(r)$.

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(3) For every real number r holds $2 \cdot \max_+(r) = r + |r|$.

(4) For all real numbers r, s such that $0 \leq r$ holds $\max_+(r \cdot s) = r \cdot \max_+(s)$.

(5) For all real numbers r, s holds $\max_+(r + s) \leq \max_+(r) + \max_+(s)$.

(6) For every real number r holds $0 \leq \max_+(r)$ and $0 \leq \max_-(r)$.

(7) For all real numbers r_1, r_2, s_1, s_2 such that $r_1 \leq s_1$ and $r_2 \leq s_2$ holds $\max(r_1, r_2) \leq \max(s_1, s_2)$.

2. Properties of Real Function

One can prove the following propositions:

(8) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all real numbers r, s such that $r \neq 0$ holds $F^{-1}\{\frac{s}{r}\} = (r\,F)^{-1}\{s\}$.

(9) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $(0\,F)^{-1}\{0\} = \operatorname{dom} F$.

(10) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $|F|^{-1}\{r\} = F^{-1}\{-r, r\}$.

(11) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $|F|^{-1}\{0\} = F^{-1}\{0\}$.

(12) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $r < 0$ holds $|F|^{-1}\{r\} = \emptyset$.

(13) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ and for every real number r such that $r \neq 0$ holds F and G are fiwerwise equipotent if and only if $r\,F$ and $r\,G$ are fiwerwise equipotent.

(14) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ holds F and G are fiwerwise equipotent if and only if $-F$ and $-G$ are fiwerwise equipotent.

(15) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $|F|$ and $|G|$ are fiwerwise equipotent.

We now define two new constructions. Let X, Y be sets. A non-empty set of functions is said to be a non empty set of partial functions from X to Y if:

(Def.3) every element of it is a partial function from X to Y.

Let X, Y be sets. Then $X \dot{\rightarrow} Y$ is a non empty set of partial functions from X to Y. Let P be a non empty set of partial functions from X to Y. We see that the element of P is a partial function from X to Y. Let D, C be non-empty sets, and let X be a subset of D, and let c be an element of C. Then $X \longmapsto c$ is an element of $D \dot{\rightarrow} C$. Let D be a non-empty set, and let F_1, F_2 be elements of $D \dot{\rightarrow} \mathbb{R}$. Then $F_1 + F_2$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Then $F_1 - F_2$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Then $F_1\,F_2$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Then $\frac{F_1}{F_2}$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Let D be a non-empty set, and let F be an element of $D \dot{\rightarrow} \mathbb{R}$. Then $|F|$ is an element of $D \dot{\to} \mathbb{R}$. Then $-F$ is an element of $D \dot{\to} \mathbb{R}$. Then $\frac{1}{F}$ is an element of $D \dot{\to} \mathbb{R}$. Let D be a non-empty set, and let F be an element of $D \dot{\to} \mathbb{R}$, and let r be a real number. Then $r\,F$ is an element of $D \dot{\to} \mathbb{R}$. Let D be a non-empty set. The functor $+_{D \dot{\to} \mathbb{R}}$ yielding a binary operation on $D \dot{\to} \mathbb{R}$ is defined as follows:

(Def.4) for all elements F_1, F_2 of $D \dot{\to} \mathbb{R}$ holds $+_{D \dot{\to} \mathbb{R}}(F_1, F_2) = F_1 + F_2$.

The following propositions are true:

(16) For every non-empty set D holds $+_{D \dot{\to} \mathbb{R}}$ is commutative.

(17) For every non-empty set D holds $+_{D \dot{\to} \mathbb{R}}$ is associative.

(18) For every non-empty set D holds $\Omega_D \longmapsto 0$ **qua** a real number is a unity w.r.t. $+_{D \dot{\to} \mathbb{R}}$.

(19) For every non-empty set D holds $\mathbf{1}_{+_{D \dot{\to} \mathbb{R}}} = \Omega_D \longmapsto 0$ **qua** a real number.

(20) For every non-empty set D holds $+_{D \dot{\to} \mathbb{R}}$ has a unity.

Let D be a non-empty set, and let f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. The functor $\sum f$ yielding an element of $D \dot{\to} \mathbb{R}$ is defined as follows:

(Def.5) $\sum f = +_{D \dot{\to} \mathbb{R}} \circledast f$.

Next we state several propositions:

(21) For every non-empty set D holds $\sum(\varepsilon_{(D \dot{\to} \mathbb{R})}) = \Omega_D \longmapsto 0$ **qua** a real number.

(22) For every non-empty set D and for every element G of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G \rangle = G$.

(23) For every non-empty set D and for every finite sequence f of elements of $D \dot{\to} \mathbb{R}$ and for every element G of $D \dot{\to} \mathbb{R}$ holds $\sum(f \frown \langle G \rangle) = \sum f + G$.

(24) For every non-empty set D and for all finite sequences f_1, f_2 of elements of $D \dot{\to} \mathbb{R}$ holds $\sum(f_1 \frown f_2) = \sum f_1 + \sum f_2$.

(25) For every non-empty set D and for every finite sequence f of elements of $D \dot{\to} \mathbb{R}$ and for every element G of $D \dot{\to} \mathbb{R}$ holds $\sum(\langle G \rangle \frown f) = G + \sum f$.

(26) For every non-empty set D and for all elements G_1, G_2 of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G_1, G_2 \rangle = G_1 + G_2$.

(27) For every non-empty set D and for all elements G_1, G_2, G_3 of $D \dot{\to} \mathbb{R}$ holds $\sum\langle G_1, G_2, G_3 \rangle = G_1 + G_2 + G_3$.

(28) For every non-empty set D and for all finite sequences f, g of elements of $D \dot{\to} \mathbb{R}$ such that f and g are fiwerwise equipotent holds $\sum f = \sum g$.

We now define four new constructions. Let D be a non-empty set, and let f be a finite sequence. The functor $\mathrm{CHI}(f, D)$ yielding a finite sequence of elements of $D \dot{\to} \mathbb{R}$ is defined by:

(Def.6) $\mathrm{len}\,\mathrm{CHI}(f, D) = \mathrm{len}\, f$ and for every n such that $n \in \mathrm{dom}\,\mathrm{CHI}(f, D)$ holds $(\mathrm{CHI}(f, D))(n) = \chi_{f(n), D}$.

Let D be a non-empty set, and let f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and let R be a finite sequence of elements of $\mathbb{R}$. The functor $R\,f$ yields a finite sequence of elements of $D \dot{\to} \mathbb{R}$ and is defined as follows:

(Def.7) $\operatorname{len}(R\,f) = \min(\operatorname{len} R, \operatorname{len} f)$ and for every n such that $n \in \operatorname{dom}(R\,f)$ and for every partial function F from D to $\mathbb{R}$ and for every r such that $r = R(n)$ and $F = f(n)$ holds $(R\,f)(n) = r\,F$.

Let D, C be non-empty sets, and let f be a finite sequence of elements of $D \dot{\rightarrow} C$, and let d be an element of D. The functor $f \# d$ yields a finite sequence of elements of C and is defined as follows:

(Def.8) $\operatorname{len}(f \# d) = \operatorname{len} f$ and for every natural number n and for every element G of $D \dot{\rightarrow} C$ such that $n \in \operatorname{dom}(f \# d)$ and $f(n) = G$ holds $(f \# d)(n) = G(d)$.

Let D, C be non-empty sets, and let f be a finite sequence of elements of $D \dot{\rightarrow} C$, and let d be an element of D. We say that d is common for dom f if and only if:

(Def.9) for every element G of $D \dot{\rightarrow} C$ and for every natural number n such that $n \in \operatorname{dom} f$ and $f(n) = G$ holds $d \in \operatorname{dom} G$.

One can prove the following propositions:

(29) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n such that d is common for dom f and $n \neq 0$ holds d is common for dom $f \upharpoonright n$.

(30) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n such that d is common for dom f holds d is common for dom $f_{\lfloor n}$.

(31) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ such that $\operatorname{len} f \neq 0$ holds d is common for dom f if and only if $d \in \operatorname{dom} \sum f$.

(32) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n holds $(f \upharpoonright n) \# d = (f \# d) \upharpoonright n$.

(33) For every non-empty set D and for every finite sequence f and for every element d of D holds d is common for dom $\operatorname{CHI}(f, D)$.

(34) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ and for every finite sequence R of elements of $\mathbb{R}$ such that d is common for dom f holds d is common for dom $R\,f$.

(35) For every non-empty set D and for every finite sequence f and for every finite sequence R of elements of $\mathbb{R}$ and for every element d of D holds d is common for dom $R\,\operatorname{CHI}(f, D)$.

(36) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ such that d is common for dom f holds $(\sum f)(d) = \sum (f \# d)$.

We now define two new functors. Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$. The functor $\max_{+}(F)$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.10) $\operatorname{dom}\max_+(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom}\max_+(F)$ holds $(\max_+(F))(d) = \max_+(F(d))$.

The functor $\max_-(F)$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.11) $\operatorname{dom}\max_-(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom}\max_-(F)$ holds $(\max_-(F))(d) = \max_-(F(d))$.

The following propositions are true:

(37) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F = \max_+(F) - \max_-(F)$ and $|F| = \max_+(F) + \max_-(F)$ and $2\max_+(F) = F + |F|$.

(38) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $F^{-1}\{r\} = (\max_+(F))^{-1}\{r\}$.

(39) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}\,]-\infty, 0] = (\max_+(F))^{-1}\{0\}$.

(40) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \operatorname{dom} F$ holds $0 \leq (\max_+(F))(d)$.

(41) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $F^{-1}\{-r\} = (\max_-(F))^{-1}\{r\}$.

(42) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}\,[0, +\infty[\, = (\max_-(F))^{-1}\{0\}$.

(43) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \operatorname{dom} F$ holds $0 \leq (\max_-(F))(d)$.

(44) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $\max_+(F)$ and $\max_+(G)$ are fiwerwise equipotent.

(45) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $\max_-(F)$ and $\max_-(G)$ are fiwerwise equipotent.

(46) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite and $\max_+(F)$ and $\max_+(G)$ are fiwerwise equipotent and $\max_-(F)$ and $\max_-(G)$ are fiwerwise equipotent holds F and G are fiwerwise equipotent.

(47) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_+(F) \restriction X = \max_+(F \restriction X)$.

(48) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_{-}(F) \restriction X = \max_{-}(F \restriction X)$.

(49) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ if for every element d of D such that $d \in \operatorname{dom} F$ holds $F(d) \geq 0$, then $\max_{+}(F) = F$.

(50) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ if for every element d of D such that $d \in \operatorname{dom} F$ holds $F(d) \leq 0$, then $\max_{-}(F) = -F$.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let r be a real number. The functor $F - r$ yields a partial function from D to $\mathbb{R}$ and is defined as follows:

(Def.12) $\operatorname{dom}(F - r) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom}(F - r)$ holds $(F - r)(d) = F(d) - r$.

We now state four propositions:

(51) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F - 0 = F$.

(52) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r and for every set X holds $F \restriction X - r = (F - r) \restriction X$.

(53) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all real numbers r, s holds $F^{-1}\{s + r\} = (F - r)^{-1}\{s\}$.

(54) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ and for every real number r holds F and G are fiwerwise equipotent if and only if $F - r$ and $G - r$ are fiwerwise equipotent.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and let X be a set. We say that F is convex on X if and only if the conditions (Def.13) is satisfied.

(Def.13) (i) $X \subseteq \operatorname{dom} F$,

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1 - p) \cdot s \in X$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

The following propositions are true:

(55) Let a, b be real numbers. Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \operatorname{dom} F$,

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in [a, b]$ and $s \in [a, b]$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

(56) Let a, b be real numbers. Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \operatorname{dom} F$,

(ii) for all real numbers x_1, x_2, x_3 such that $x_1 \in [a,b]$ and $x_2 \in [a,b]$ and $x_3 \in [a,b]$ and $x_1 < x_2$ and $x_2 < x_3$ holds $\frac{F(x_1)-F(x_2)}{x_1-x_2} \leq \frac{F(x_2)-F(x_3)}{x_2-x_3}$.

(57) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for all sets X, Y such that F is convex on X and $Y \subseteq X$ holds F is convex on Y.

(58) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r holds F is convex on X if and only if $F - r$ is convex on X.

(59) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r such that $0 < r$ holds F is convex on X if and only if $r\,F$ is convex on X.

(60) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that $X \subseteq \operatorname{dom} F$ holds $0\,F$ is convex on X.

(61) For all partial functions F, G from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that F is convex on X and G is convex on X holds $F + G$ is convex on X.

(62) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r such that F is convex on X holds $\max_{+}(F - r)$ is convex on X.

(63) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that F is convex on X holds $\max_{+}(F)$ is convex on X.

(64) $\mathrm{id}_{(\Omega_{\mathbb{R}})}$ is convex on $\mathbb{R}$.

(65) For every real number r holds $\max_{+}(\mathrm{id}_{(\Omega_{\mathbb{R}})} - r)$ is convex on $\mathbb{R}$.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let X be a set. Let us assume that $\operatorname{dom}(F \restriction X)$ is finite. The functor $\operatorname{FinS}(F, X)$ yields a non-increasing finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.14) $F \restriction X$ and $\operatorname{FinS}(F, X)$ are fiwerwise equipotent.

Next we state a number of propositions:

(66) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{FinS}(F, \operatorname{dom}(F \restriction X)) = \operatorname{FinS}(F, X)$.

(67) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{FinS}(F \restriction X, X) = \operatorname{FinS}(F, X)$.

(68) For every non-empty set D and for every element d of D and for every set X and for every partial function F from D to $\mathbb{R}$ such that X is finite and $d \in \operatorname{dom}(F \restriction X)$ holds $(\operatorname{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiwerwise equipotent.

(69) For every non-empty set D and for every element d of D and for every set X and for every partial function F from D to $\mathbb{R}$ such that $\operatorname{dom}(F \restriction X)$ is finite and $d \in \operatorname{dom}(F \restriction X)$ holds $(\operatorname{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiwerwise equipotent.

(70) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{len} \operatorname{FinS}(F, X) =$

card dom$(F \restriction X)$.

(71) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $\mathrm{FinS}(F, \emptyset) = \varepsilon_{\mathbb{R}}$.

(72) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \mathrm{dom}\, F$ holds $\mathrm{FinS}(F, \{d\}) = \langle F(d) \rangle$.

(73) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Then for every set X and for every element d of D such that $\mathrm{dom}(F \restriction X)$ is finite and $d \in \mathrm{dom}(F \restriction X)$ and $(\mathrm{FinS}(F, X))(\mathrm{len}\,\mathrm{FinS}(F, X)) = F(d)$ holds $\mathrm{FinS}(F, X) = (\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$.

(74) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Let X, Y be sets. Suppose $\mathrm{dom}(F \restriction X)$ is finite and $Y \subseteq X$ and for all elements d_1, d_2 of D such that $d_1 \in \mathrm{dom}(F \restriction Y)$ and $d_2 \in \mathrm{dom}(F \restriction (X \setminus Y))$ holds $F(d_1) \geq F(d_2)$. Then $\mathrm{FinS}(F, X) = (\mathrm{FinS}(F, Y)) \frown \mathrm{FinS}(F, X \setminus Y)$.

(75) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Let r be a real number. Let X be a set. Then for every element d of D such that $\mathrm{dom}(F \restriction X)$ is finite and $d \in \mathrm{dom}(F \restriction X)$ holds
$(\mathrm{FinS}(F - r, X))(\mathrm{len}\,\mathrm{FinS}(F - r, X)) = (F - r)(d)$
if and only if $(\mathrm{FinS}(F, X))(\mathrm{len}\,\mathrm{FinS}(F, X)) = F(d)$.

(76) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r and for every set X such that $\mathrm{dom}(F \restriction X)$ is finite holds $\mathrm{FinS}(F - r, X) = \mathrm{FinS}(F, X) - (\mathrm{card}\,\mathrm{dom}(F \restriction X) \longmapsto r)$.

(77) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\mathrm{dom}(F \restriction X)$ is finite and for every element d of D such that $d \in \mathrm{dom}(F \restriction X)$ holds $F(d) \geq 0$ holds $\mathrm{FinS}(\mathrm{max}_{+}(F), X) = \mathrm{FinS}(F, X)$.

(78) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\mathrm{dom}(F \restriction X)$ is finite and $\mathrm{rng}(F \restriction X) = \{r\}$ holds $\mathrm{FinS}(F, X) = \mathrm{card}\,\mathrm{dom}(F \restriction X) \longmapsto r$.

(79) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and $X \cap Y = \emptyset$ holds $\mathrm{FinS}(F, X \cup Y)$ and $(\mathrm{FinS}(F, X)) \frown \mathrm{FinS}(F, Y)$ are fiwerwise equipotent.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let X be a set. The functor $\sum_{\kappa=0}^{X} F(\kappa)$ yields a real number and is defined as follows:

(Def.15) $\sum_{\kappa=0}^{X} F(\kappa) = \sum \mathrm{FinS}(F, X)$.

One can prove the following propositions:

(80) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\mathrm{dom}(F \restriction X)$ is finite holds $\sum_{\kappa=0}^{X} (r\, F)(\kappa) = r \cdot \sum_{\kappa=0}^{X} F(\kappa)$.

(81) For every non-empty set D and for all partial functions F, G from D to

$\mathbb{R}$ and for every set X such that $\mathrm{dom}(F \upharpoonright X)$ is finite and $\mathrm{dom}(F \upharpoonright X) = \mathrm{dom}(G \upharpoonright X)$ holds $\sum_{\kappa=0}^{X}(F+G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{X} G(\kappa)$.

(82) For every non-empty set D and for all partial functions F, G from D to $\mathbb{R}$ and for every set X such that $\mathrm{dom}(F \upharpoonright X)$ is finite and $\mathrm{dom}(F \upharpoonright X) = \mathrm{dom}(G \upharpoonright X)$ holds $\sum_{\kappa=0}^{X}(F-G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - \sum_{\kappa=0}^{X} G(\kappa)$.

(83) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\mathrm{dom}(F \upharpoonright X)$ is finite holds $\sum_{\kappa=0}^{X}(F-r)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - r \cdot \mathrm{card}\,\mathrm{dom}(F \upharpoonright X)$.

(84) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $\sum_{\kappa=0}^{\emptyset} F(\kappa) = 0$.

(85) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \mathrm{dom}\,F$ holds $\sum_{\kappa=0}^{\{d\}} F(\kappa) = F(d)$.

(86) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\mathrm{dom}(F \upharpoonright (X \cup Y))$ is finite and $X \cap Y = \emptyset$ holds $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

(87) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\mathrm{dom}(F \upharpoonright (X \cup Y))$ is finite and $\mathrm{dom}(F \upharpoonright X) \cap \mathrm{dom}(F \upharpoonright Y) = \emptyset$ holds $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

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