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The Lattice of Domains of an Extremally Disconnected Space [1]

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Summary. Let X be a topological space and let A be a subset of X. Recall that A is said to be a *domain* in X provided $\operatorname{Int}\overline{A} \subseteq A \subseteq \overline{\operatorname{Int} A}$ (see [24], [11]). Recall also that A is said to be a(n) *closed* (*open*) *domain* in X if $A = \overline{\operatorname{Int} A}$ ($A = \operatorname{Int}\overline{A}$, resp.) (see e.g. [14], [24]). It is well-known that for a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too (see e.g., [15], [3]). In [23] it is proved that all domains of a given topological space form a complemented lattice. One may ask whether the lattice of all domains is Boolean. The aim is to give an answer to this question.

To present the main results we first recall the definition of a class of topological spaces which is important here. X is called *extremally disconnected* if for every open subset A of X the closure $\overline{A}$ is open in X [18] (comp. [10]). It is shown here, using Mizar System, that *the lattice of all domains of a topological space X is modular iff X is extremally disconnected.* Moreover, for every extremally disconnected space the lattice of all its domains coincides with both the lattice of all its closed domains and the lattice of all its open domains. From these facts it follows that *the lattice of all domains of a topological space X is Boolean iff X is extremally disconnected.*

Note that we also review some of the standard facts on discrete, anti-discrete, almost discrete, extremally disconnected and hereditarily extremally disconnected topological spaces (comp. [14], [10]).

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The notation and terminology used in this paper are introduced in the following articles: [20], [22], [21], [16], [6], [7], [17], [24], [9], [4], [19], [12], [5], [25], [8], [2], [1], [23], and [13].

[1] *Editor's Note:* This work has won the 1992 Śleszyński's Award of the Mizar Society.

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1. Selected Properties of Subsets of a Topological Space

In the sequel X will be a topological space. We now state the proposition

(1) For every set B and for every subset A of X such that $B \subseteq A$ holds B is a subset of X.

In the sequel C denotes a subset of X. We now state three propositions:

(2) $\overline{C} = (\mathrm{Int}(C^{\mathrm{c}}))^{\mathrm{c}}$.

(3) $\overline{C^{\mathrm{c}}} = (\mathrm{Int}\, C)^{\mathrm{c}}$.

(4) $\mathrm{Int}(C^{\mathrm{c}}) = \overline{C}^{\mathrm{c}}$.

In the sequel A, B denote subsets of X. Next we state several propositions:

(5) If $A \cap B = \emptyset$, then if A is open, then $A \cap \overline{B} = \emptyset$ and also if B is open, then $\overline{A} \cap B = \emptyset$.

(6) If $A \cup B =$ the carrier of X, then if A is closed, then $A \cup \mathrm{Int}\, B =$ the carrier of X and also if B is closed, then $\mathrm{Int}\, A \cup B =$ the carrier of X.

(7) A is open and A is closed if and only if $\overline{A} = \mathrm{Int}\, A$.

(8) If A is open and A is closed, then $\mathrm{Int}\, \overline{A} = \overline{\mathrm{Int}\, A}$.

(9) If A is a domain and $\overline{\mathrm{Int}\, A} \subseteq \mathrm{Int}\, \overline{A}$, then A is an open domain and A is a closed domain.

(10) If A is a domain and $\overline{\mathrm{Int}\, A} \subseteq \mathrm{Int}\, \overline{A}$, then A is open and A is closed.

(11) If A is a domain, then $\mathrm{Int}\, \overline{A} = \mathrm{Int}\, A$ and $\overline{A} = \overline{\mathrm{Int}\, A}$.

2. Discrete Topological Structures

We now define two new attributes. A topological structure is discrete if:

(Def.1) the topology of it $= 2^{\text{the carrier of it}}$.

A topological structure is anti-discrete if:

(Def.2) the topology of it $= \{\emptyset, \text{the carrier of it}\}$.

Next we state two propositions:

(12) For every Y being a topological structure such that Y is discrete and Y is anti-discrete holds $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$.

(13) For every Y being a topological structure such that $\emptyset \in$ the topology of Y and the carrier of $Y \in$ the topology of Y holds if $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$, then Y is discrete and Y is anti-discrete.

Let us mention that there exists a topological structure which is discrete anti-discrete and strict.

Next we state two propositions:

(14) For every Y being a discrete topological structure and for every subset A of the carrier of Y holds (the carrier of Y) $\setminus A \in$ the topology of Y.

(15) For every Y being an anti-discrete topological structure and for every subset A of the carrier of Y such that $A \in$ the topology of Y holds (the carrier of Y) $\setminus A \in$ the topology of Y.

Let us observe that every topological structure which is discrete is also topological space-like and every anti-discrete topological structure is topological space-like.

One can prove the following proposition

(16) For every Y being a topological space-like topological structure such that $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$ holds Y is discrete and Y is anti-discrete.

A topological structure is almost discrete if:

(Def.3) for every subset A of the carrier of it such that $A \in$ the topology of it holds (the carrier of it) $\setminus A \in$ the topology of it.

One can verify the following observations:

* every topological structure which is discrete is also almost discrete,
* every topological structure which is anti-discrete is also almost discrete, and
* there exists an almost discrete strict topological structure.

3. Discrete Topological Spaces

Let us mention that there exists a discrete anti-discrete strict topological space.

In the sequel X denotes a topological space. Next we state three propositions:

(17) X is discrete if and only if every subset of X is open.

(18) X is discrete if and only if every subset of X is closed.

(19) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is open, then X is discrete.

Let X be a discrete topological space. Note that every subspace of X is open closed and discrete.

Let X be a discrete topological space. Observe that there exists a discrete strict subspace of X.

Next we state three propositions:

(20) X is anti-discrete if and only if for every subset A of X such that A is open holds $A = \emptyset$ or $A =$ the carrier of X.

(21) X is anti-discrete if and only if for every subset A of X such that A is closed holds $A = \emptyset$ or $A =$ the carrier of X.

(22) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A} =$ the carrier of X, then X is anti-discrete.

Let X be an anti-discrete topological space. Observe that every subspace of X is anti-discrete.

Let X be an anti-discrete topological space. Note that there exists an anti-discrete subspace of X.

One can prove the following propositions:

(23) X is almost discrete if and only if for every subset A of X such that A is open holds A is closed.

(24) X is almost discrete if and only if for every subset A of X such that A is closed holds A is open.

(25) X is almost discrete if and only if for every subset A of X such that A is open holds $\overline{A} = A$.

(26) X is almost discrete if and only if for every subset A of X such that A is closed holds $\mathrm{Int}\, A = A$.

Let us observe that there exists an almost discrete strict topological space.

One can prove the following two propositions:

(27) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A}$ is open, then X is almost discrete.

(28) X is discrete if and only if X is almost discrete and for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is closed.

Let us observe that every discrete topological space is almost discrete and every anti-discrete topological space is almost discrete.

Let X be an almost discrete topological space. Observe that every subspace of X is almost discrete.

Let X be an almost discrete topological space. One can verify that every open subspace of X is closed and every closed subspace of X is open.

Let X be an almost discrete topological space. Note that there exists a subspace of X which is almost discrete and strict.

4. Extremally Disconnected Topological Spaces

A topological space is extremally disconnected if:

(Def.4) for every subset A of it such that A is open holds $\overline{A}$ is open.

Let us note that there exists a topological space which is extremally disconnected and strict.

In the sequel X denotes a topological space. The following propositions are true:

(29) X is extremally disconnected if and only if for every subset A of X such that A is closed holds $\mathrm{Int}\, A$ is closed.

(30) X is extremally disconnected if and only if for all subsets A, B of X such that A is open and B is open holds if $A \cap B = \emptyset$, then $\overline{A} \cap \overline{B} = \emptyset$.

(31) X is extremally disconnected if and only if for all subsets A, B of X such that A is closed and B is closed holds if $A \cup B =$ the carrier of X, then $\mathrm{Int}\, A \cup \mathrm{Int}\, B =$ the carrier of X.

(32) X is extremally disconnected if and only if for every subset A of X such that A is open holds $\overline{A} = \operatorname{Int} \overline{A}$.

(33) X is extremally disconnected if and only if for every subset A of X such that A is closed holds $\operatorname{Int} A = \overline{\operatorname{Int} A}$.

(34) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds A is closed and A is open.

(35) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds A is a closed domain and A is an open domain.

(36) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds $\operatorname{Int} \overline{A} = \overline{\operatorname{Int} A}$.

(37) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds $\operatorname{Int} A = \overline{A}$.

(38) X is extremally disconnected if and only if for every subset A of X holds if A is an open domain, then A is a closed domain and also if A is a closed domain, then A is an open domain.

A topological space is hereditarily extremally disconnected if:

(Def.5) every subspace of it is extremally disconnected.

One can check the following observations:

* there exists a hereditarily extremally disconnected strict topological space,
* every hereditarily extremally disconnected topological space is extremally disconnected, and
* every topological space which is almost discrete is also hereditarily extremally disconnected.

One can prove the following proposition

(39) For every extremally disconnected topological space X and for every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 and A is dense holds X_0 is extremally disconnected.

Let X be an extremally disconnected topological space. One can check that every open subspace of X is extremally disconnected.

Let X be an extremally disconnected topological space. Note that there exists an extremally disconnected strict subspace of X.

Let X be a hereditarily extremally disconnected topological space. Note that every subspace of X is hereditarily extremally disconnected.

Let X be a hereditarily extremally disconnected topological space. Note that there exists a hereditarily extremally disconnected strict subspace of X.

One can prove the following proposition

(40) If every closed subspace of X is extremally disconnected, then X is hereditarily extremally disconnected.

5. The Lattice of Domains of Extremally Disconnected Spaces

In the sequel Y is an extremally disconnected topological space. The following propositions are true:

(41) The domains of Y = the closed domains of Y.

(42) D-Union(Y) = CLD-Union(Y) and D-Meet(Y) = CLD-Meet(Y).

(43) The lattice of domains of Y = the lattice of closed domains of Y.

(44) The domains of Y = the open domains of Y.

(45) D-Union(Y) = OPD-Union(Y) and D-Meet(Y) = OPD-Meet(Y).

(46) The lattice of domains of Y = the lattice of open domains of Y.

(47) For all elements A, B of the domains of Y holds (D-Union(Y))(A, B) = $A \cup B$ and (D-Meet(Y))(A, B) = $A \cap B$.

(48) For all elements a, b of the lattice of domains of Y and for all elements A, B of the domains of Y such that $a = A$ and $b = B$ holds $a \sqcup b = A \cup B$ and $a \sqcap b = A \cap B$.

(49) For every family F of subsets of Y such that F is domains-family and for every subset S of the lattice of domains of Y such that $S = F$ holds $\bigsqcup_{(\text{the lattice of domains of } Y)} S = \overline{\bigcup F}$.

(50) For every family F of subsets of Y such that F is domains-family and for every subset S of the lattice of domains of Y such that $S = F$ holds if $S \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \operatorname{Int} \bigcap F$ and also if $S = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \Omega_Y$.

In the sequel X will denote a topological space. One can prove the following propositions:

(51) X is extremally disconnected if and only if the lattice of domains of X is a modular lattice.

(52) If the lattice of domains of X = the lattice of closed domains of X, then X is extremally disconnected.

(53) If the lattice of domains of X = the lattice of open domains of X, then X is extremally disconnected.

(54) If the lattice of closed domains of X = the lattice of open domains of X, then X is extremally disconnected.

(55) X is extremally disconnected if and only if the lattice of domains of X is a Boolean lattice.

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