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On Discrete and Almost Discrete Topological Spaces

Zbigniew Karno
Warsaw University
Białystok

Summary. A topological space X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open (comp. [6],[5]). Almost discrete spaces were investigated in Mizar formalism in [2]. We present here a few properties of such spaces supplementary to those given in [2].

Most interesting is the following characterization : *A topological space X is almost discrete iff every nonempty subset of X is not nowhere dense.* Hence, *X is non almost discrete iff there is an everywhere dense subset of X different from the carrier of X*. We have an analogous characterization of discrete spaces : *A topological space X is discrete iff every nonempty subset of X is not boundary.* Hence, *X is non discrete iff there is a dense subset of X different from the carrier of X*. It is well known that the class of all almost discrete spaces contains both the class of discrete spaces and the class of anti-discrete spaces (see e.g., [2]). Observations presented here show that the class of all almost discrete non discrete spaces is not contained in the class of anti-discrete spaces and the class of all almost discrete non anti-discrete spaces is not contained in the class of discrete spaces. Moreover, the class of almost discrete non discrete non anti-discrete spaces is nonempty. To analyse these interdependencies we use various examples of topological spaces constructed here in Mizar formalism.

MML Identifier: `TEX_1`.

The papers [12], [14], [9], [11], [7], [13], [8], [15], [10], [4], [1], [2], and [3] provide the notation and terminology for this paper.

1. Properties of Subsets of a Topological Space with Modified Topology

In the sequel X will be a topological space and D will be a subset of X. One can prove the following propositions:

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(1) For every subset B of X and for every subset C of the X modified w.r.t. D such that $B = C$ holds if B is open, then C is open.

(2) For every subset B of X and for every subset C of the X modified w.r.t. D such that $B = C$ holds if B is closed, then C is closed.

(3) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C = D$ holds C is closed.

(4) For every subset C of the X modified w.r.t. D such that $C = D$ holds if D is dense, then C is dense and C is open.

(5) For every subset C of the X modified w.r.t. D such that $D \subseteq C$ holds if D is dense, then C is everywhere dense.

(6) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C = D$ holds if D is boundary, then C is boundary and C is closed.

(7) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C \subseteq D$ holds if D is boundary, then C is nowhere dense.

2. Trivial Topological Spaces

Let us observe that a 1-sorted structure is trivial if:

(Def.1) there exists an element d of the carrier of it such that the carrier of it $= \{d\}$.

One can verify the following observations:

* there exists a 1-sorted structure which is trivial and strict,
* there exists a 1-sorted structure which is non trivial and strict,
* there exists a topological structure which is trivial and strict, and
* there exists a non trivial strict topological structure.

One can prove the following proposition

(8) For every Y being a trivial topological structure such that the topology of Y is non-empty holds if Y is almost discrete, then Y is topological space-like.

One can verify the following observations:

* there exists a trivial strict topological space,
* every topological space which is trivial is also anti-discrete and discrete,
* every discrete anti-discrete topological space is trivial,
* there exists a topological space which is non trivial and strict,
* every non discrete topological space is non trivial, and
* every non anti-discrete topological space is non trivial.

3. Examples of Discrete and Anti-discrete Topological Spaces

We now define two new functors. Let D be a set. The functor 2_*^D yielding a non-empty family of subsets of D is defined by:

(Def.2) $2_*^D = \{\emptyset, D\}$.

Let D be a non-empty set. The functor ADTS(D) yields an anti-discrete strict topological space and is defined as follows:

(Def.3) $\mathrm{ADTS}(D) = \langle D, 2_*^D \rangle$.

We now state several propositions:

(9) For every anti-discrete topological space X holds the topological structure of X = ADTS(the carrier of X).

(10) For every topological space X such that the topological structure of X = the topological structure of ADTS(the carrier of X) holds X is anti-discrete.

(11) For every anti-discrete topological space X and for every subset A of X holds if A is empty, then $\overline{A} = \emptyset$ and also if A is non-empty, then $\overline{A}$ = the carrier of X.

(12) For every anti-discrete topological space X and for every subset A of X holds if $A \neq$ the carrier of X, then $\mathrm{Int}\, A = \emptyset$ and also if A = the carrier of X, then $\mathrm{Int}\, A$ = the carrier of X.

(13) For every topological space X if for every subset A of X such that A is non-empty holds $\overline{A}$ = the carrier of X, then X is anti-discrete.

(14) For every topological space X if for every subset A of X such that $A \neq$ the carrier of X holds $\mathrm{Int}\, A = \emptyset$, then X is anti-discrete.

(15) For every anti-discrete topological space X and for every subset A of X holds if $A \neq \emptyset$, then A is dense and also if $A \neq$ the carrier of X, then A is boundary.

(16) For every topological space X if for every subset A of X such that $A \neq \emptyset$ holds A is dense, then X is anti-discrete.

(17) For every topological space X if for every subset A of X such that $A \neq$ the carrier of X holds A is boundary, then X is anti-discrete.

Let D be a set. Then 2^D is a non-empty family of subsets of D. Let D be a non-empty set. The functor DTS(D) yielding a discrete strict topological space is defined by:

(Def.4) $\mathrm{DTS}(D) = \langle D, 2^D \rangle$.

One can prove the following propositions:

(18) For every discrete topological space X holds the topological structure of X = DTS(the carrier of X).

(19) For every topological space X such that the topological structure of X = the topological structure of DTS(the carrier of X) holds X is discrete.

(20) For every discrete topological space X and for every subset A of X holds $\overline{A} = A$ and $\mathrm{Int}\, A = A$.

(21) For every topological space X if for every subset A of X holds $\overline{A} = A$, then X is discrete.

(22) For every topological space X if for every subset A of X holds $\mathrm{Int}\, A = A$, then X is discrete.

(23) For every non-empty set D holds $\mathrm{ADTS}(D) = \mathrm{DTS}(D)$ if and only if there exists an element d_0 of D such that $D = \{d_0\}$.

Let us note that there exists a discrete non anti-discrete strict topological space and there exists an anti-discrete non discrete strict topological space.

4. An Example of a Topological Space

Let D be a set, and let F be a family of subsets of D, and let S be a set. Then $F \setminus S$ is a family of subsets of D. Let D be a non-empty set, and let d_0 be an element of D. The functor $\mathrm{STS}(D, d_0)$ yields a strict topological space and is defined as follows:

(Def.5) $\mathrm{STS}(D, d_0) = \langle D, 2^D \setminus \{A : d_0 \in A \wedge A \neq D\}\rangle$, where A ranges over subsets of D.

In the sequel D denotes a non-empty set and d_0 denotes an element of D. One can prove the following propositions:

(24) For every subset A of $\mathrm{STS}(D, d_0)$ holds if $\{d_0\} \subseteq A$, then A is closed and also if A is non-empty and A is closed, then $\{d_0\} \subseteq A$.

(25) If $D \setminus \{d_0\}$ is non-empty, then for every subset A of $\mathrm{STS}(D, d_0)$ holds if $A = \{d_0\}$, then A is closed and A is boundary and also if A is non-empty and A is closed and A is boundary, then $A = \{d_0\}$.

(26) For every subset A of $\mathrm{STS}(D, d_0)$ holds if $A \subseteq D \setminus \{d_0\}$, then A is open and also if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(27) If $D \setminus \{d_0\}$ is non-empty, then for every subset A of $\mathrm{STS}(D, d_0)$ holds if $A = D \setminus \{d_0\}$, then A is open and A is dense and also if $A \neq D$ and A is open and A is dense, then $A = D \setminus \{d_0\}$.

Let us observe that there exists a non anti-discrete non discrete strict topological space.

The following propositions are true:

(28) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y holds if $\{d_0\} \subseteq A$, then A is closed and also if A is non-empty and A is closed, then $\{d_0\} \subseteq A$.

(29) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y holds if $A \subseteq D \setminus \{d_0\}$, then A is open and also if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(30) For every topological space Y holds the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every non-empty subset A of Y holds $\overline{A} = A \cup \{d_0\}$.

(31) For every topological space Y holds the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y such that $A \neq D$ holds $\mathrm{Int}\, A = A \setminus \{d_0\}$.

(32) $\mathrm{STS}(D, d_0) = \mathrm{ADTS}(D)$ if and only if $D = \{d_0\}$.

(33) $\mathrm{STS}(D, d_0) = \mathrm{DTS}(D)$ if and only if $D = \{d_0\}$.

(34) For every non-empty set D and for every element d_0 of D and for every subset A of $\mathrm{STS}(D, d_0)$ such that $A = \{d_0\}$ holds $\mathrm{DTS}(D)$ = the $\mathrm{STS}(D, d_0)$ modified w.r.t. A.

5. Discrete and Almost Discrete Spaces

Let us observe that a topological space is discrete if:

(Def.6) for every non-empty subset A of it holds A is not boundary.

We now state the proposition

(35) X is discrete if and only if for every subset A of X such that $A \neq$ the carrier of X holds A is not dense.

One can verify that every non almost discrete topological space is non discrete and non anti-discrete.

Let us observe that a topological space is almost discrete if:

(Def.7) for every non-empty subset A of it holds A is not nowhere dense.

Next we state three propositions:

(36) X is almost discrete if and only if for every subset A of X such that $A \neq$ the carrier of X holds A is everywhere dense.

(37) X is non almost discrete if and only if there exists a non-empty subset A of X such that A is boundary and A is closed.

(38) X is non almost discrete if and only if there exists a subset A of X such that $A \neq$ the carrier of X and A is dense and A is open.

One can verify that there exists an almost discrete non discrete non anti-discrete strict topological space.

Next we state the proposition

(39) For every non-empty set C and for every element c_0 of C holds $C \setminus \{c_0\}$ is non-empty if and only if $\mathrm{STS}(C, c_0)$ is non almost discrete.

Let us observe that there exists a non almost discrete strict topological space.

We now state two propositions:

(40) For every non-empty subset A of X such that A is boundary holds the X modified w.r.t. A^{c} is non almost discrete.

(41) For every subset A of X such that $A \neq$ the carrier of X and A is dense holds the X modified w.r.t. A is non almost discrete.

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References

[1] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.
[2] Zbigniew Karno. The lattice of domains of an extremally disconnected space. *Formalized Mathematics*, 3(**2**):143–149, 1992.
[3] Zbigniew Karno. Remarks on special subsets of topological spaces. *Formalized Mathematics*, 3(**2**):297–303, 1992.
[4] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.
[5] Kazimierz Kuratowski. *Topology*. Volume II, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1968.
[6] Kazimierz Kuratowski. *Topology*. Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[7] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[8] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[9] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[10] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[11] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[15] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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