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Remarks on Special Subsets of Topological Spaces

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Summary. Let X be a topological space and let A be a subset of X. Recall that A is *nowhere dense* in X if its closure is a boundary subset of X, i.e., if $\mathrm{Int}\,\overline{A} = \emptyset$ (see [2]). We introduce here the concept of everywhere dense subsets in X, which is dual to the above one. Namely, A is said to be *everywhere dense* in X if its interior is a dense subset of X, i.e., if $\overline{\mathrm{Int}\,A} =$ the carrier of X.

Our purpose is to list a number of properties of such sets (comp. [7]). As a sample we formulate their two dual characterizations. The first one characterizes thin sets in X : *A is nowhere dense iff for every open nonempty subset G of X there is an open nonempty subset of X contained in G and disjoint from A*. The corresponding second one characterizes thick sets in X : *A is everywhere dense iff for every closed subset F of X distinct from the carrier of X there is a closed subset of X distinct from the carrier of X, which contains F and together with A covers the carrier of X*. We also give some connections between both these concepts. Of course, *A is everywhere (nowhere) dense in X iff its complement is nowhere (everywhere) dense*. Moreover, *A is nowhere dense iff there are two subsets of X, C boundary closed and B everywhere dense, such that $A = C \cap B$ and $C \cup B$ covers the carrier of X*. Dually, *A is everywhere dense iff there are two disjoint subsets of X, C open dense and B nowhere dense, such that $A = C \cup B$*.

Note that some relationships between everywhere (nowhere) dense sets in X and everywhere (nowhere) dense sets in subspaces of X are also indicated.

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The notation and terminology used here are introduced in the following papers: [5], [6], [3], [7], [4], and [1].

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1. Selected Properties of Subsets of a Topological Space

In the sequel X will denote a topological space and A, B will denote subsets of X. We now state several propositions:

(1) $A = \emptyset_X$ if and only if $A^c = \Omega_X$ and also $A = \emptyset$ if and only if $A^c =$ the carrier of X.

(2) $A = \Omega_X$ if and only if $A^c = \emptyset_X$ and also $A =$ the carrier of X if and only if $A^c = \emptyset$.

(3) $\operatorname{Int} A \cap \overline{B} \subseteq \overline{A \cap B}$.

(4) $\operatorname{Int}(A \cup B) \subseteq \overline{A} \cup \operatorname{Int} B$.

(5) If A is closed, then $\operatorname{Int}(A \cup B) \subseteq A \cup \operatorname{Int} B$.

(6) If A is closed, then $\operatorname{Int}(A \cup B) = \operatorname{Int}(A \cup \operatorname{Int} B)$.

(7) If $A \cap \operatorname{Int} \overline{A} = \emptyset$, then $\operatorname{Int} \overline{A} = \emptyset$.

(8) If $A \cup \overline{\operatorname{Int} A} =$ the carrier of X, then $\overline{\operatorname{Int} A} =$ the carrier of X.

2. Special Subsets of a Topological Space

Let X be a topological space. Let us observe that a subset of X is boundary if:

(Def.1) $\operatorname{Int} \mathrm{it} = \emptyset$.

We now state several propositions:

(9) $\emptyset_X$ is boundary.

(10) If A is boundary, then $A \neq$ the carrier of X.

(11) If B is boundary and $A \subseteq B$, then A is boundary.

(12) A is boundary if and only if for every subset C of X such that $A^c \subseteq C$ and C is closed holds $C =$ the carrier of X.

(13) A is boundary if and only if for every subset G of X such that $G \neq \emptyset$ and G is open holds $A^c \cap G \neq \emptyset$.

(14) A is boundary if and only if for every subset F of X such that F is closed holds $\operatorname{Int} F = \operatorname{Int}(F \cup A)$.

(15) If A is boundary or B is boundary, then $A \cap B$ is boundary.

Let X be a topological space. Let us observe that a subset of X is dense if:

(Def.2) $\overline{\mathrm{it}} =$ the carrier of X.

Next we state several propositions:

(16) Ω_X is dense.

(17) If A is dense, then $A \neq \emptyset$.

(18) A is dense if and only if A^c is boundary.

(19) A is dense if and only if for every subset C of X such that $A \subseteq C$ and C is closed holds $C =$ the carrier of X.

(20) A is dense if and only if for every subset G of X such that G is open holds $\overline{G} = \overline{G \cap A}$.

(21) If A is dense or B is dense, then $A \cup B$ is dense.

Let X be a topological space. Let us observe that a subset of X is nowhere dense if:

(Def.3) $\mathrm{Int}\,\overline{\mathrm{it}} = \emptyset$.

The following propositions are true:

(22) $\emptyset_X$ is nowhere dense.

(23) If A is nowhere dense, then $A \neq$ the carrier of X.

(24) If A is nowhere dense, then $\overline{A}$ is nowhere dense.

(25) If A is nowhere dense, then A is not dense.

(26) If B is nowhere dense and $A \subseteq B$, then A is nowhere dense.

(27) A is nowhere dense if and only if there exists a subset C of X such that $A \subseteq C$ and C is closed and C is boundary.

(28) A is nowhere dense if and only if for every subset G of X such that $G \neq \emptyset$ and G is open there exists a subset H of X such that $H \subseteq G$ and $H \neq \emptyset$ and H is open and $A \cap H = \emptyset$.

(29) If A is nowhere dense or B is nowhere dense, then $A \cap B$ is nowhere dense.

(30) If A is nowhere dense and B is boundary, then $A \cup B$ is boundary.

Let X be a topological space. A subset of X is everywhere dense if:

(Def.4) $\overline{\mathrm{Int}\,\mathrm{it}} = \Omega_X$.

Let X be a topological space. Let us observe that a subset of X is everywhere dense if:

(Def.5) $\overline{\mathrm{Int}\,\mathrm{it}} =$ the carrier of X.

One can prove the following propositions:

(31) Ω_X is everywhere dense.

(32) If A is everywhere dense, then $\mathrm{Int}\,A$ is everywhere dense.

(33) If A is everywhere dense, then A is dense.

(34) If A is everywhere dense, then $A \neq \emptyset$.

(35) A is everywhere dense if and only if $\mathrm{Int}\,A$ is dense.

(36) If A is open and A is dense, then A is everywhere dense.

(37) If A is everywhere dense, then A is not boundary.

(38) If A is everywhere dense and $A \subseteq B$, then B is everywhere dense.

(39) A is everywhere dense if and only if A^{c} is nowhere dense.

(40) A is nowhere dense if and only if A^{c} is everywhere dense.

(41) A is everywhere dense if and only if there exists a subset C of X such that $C \subseteq A$ and C is open and C is dense.

(42) A is everywhere dense if and only if for every subset F of X such that $F \neq$ the carrier of X and F is closed there exists a subset H of X such that $F \subseteq H$ and $H \neq$ the carrier of X and H is closed and $A \cup H =$ the carrier of X.

(43) If A is everywhere dense or B is everywhere dense, then $A \cup B$ is everywhere dense.

(44) If A is everywhere dense and B is everywhere dense, then $A \cap B$ is everywhere dense.

(45) If A is everywhere dense and B is dense, then $A \cap B$ is dense.

(46) If A is dense and B is nowhere dense, then $A \setminus B$ is dense.

(47) If A is everywhere dense and B is boundary, then $A \setminus B$ is dense.

(48) If A is everywhere dense and B is nowhere dense, then $A \setminus B$ is everywhere dense.

In the sequel D denotes a subset of X. We now state four propositions:

(49) If D is everywhere dense, then there exist subsets C, B of X such that C is open and C is dense and B is nowhere dense and $C \cup B = D$ and $C \cap B = \emptyset$.

(50) If D is everywhere dense, then there exist subsets C, B of X such that C is open and C is dense and B is closed and B is boundary and $C \cup D \cap B = D$ and $C \cap B = \emptyset$ and $C \cup B =$ the carrier of X.

(51) If D is nowhere dense, then there exist subsets C, B of X such that C is closed and C is boundary and B is everywhere dense and $C \cap B = D$ and $C \cup B =$ the carrier of X.

(52) If D is nowhere dense, then there exist subsets C, B of X such that C is closed and C is boundary and B is open and B is dense and $C \cap (D \cup B) = D$ and $C \cap B = \emptyset$ and $C \cup B =$ the carrier of X.

3. Properties of Subsets in Subspaces

In the sequel Y_0 will denote a subspace of X. One can prove the following propositions:

(53) For every subset A of X and for every subset B of Y_0 such that $B \subseteq A$ holds $\overline{B} \subseteq \overline{A}$.

(54) For all subsets C, A of X and for every subset B of Y_0 such that C is closed and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$ holds $\overline{A} = \overline{B}$.

(55) For every closed subspace Y_0 of X and for every subset A of X and for every subset B of Y_0 such that $A = B$ holds $\overline{A} = \overline{B}$.

(56) For every subset A of X and for every subset B of Y_0 such that $A \subseteq B$ holds $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(57) For all subsets C, A of X and for every subset B of Y_0 such that C is open and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$ holds $\operatorname{Int} A = \operatorname{Int} B$.

(58) For every open subspace Y_0 of X and for every subset A of X and for every subset B of Y_0 such that $A = B$ holds $\operatorname{Int} A = \operatorname{Int} B$.

In the sequel X_0 denotes a subspace of X. The following propositions are true:

(59) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if A is dense, then B is dense.

(60) For all subsets C, A of X and for every subset B of X_0 such that $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is dense and B is dense if and only if A is dense.

(61) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if A is everywhere dense, then B is everywhere dense.

(62) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is dense and B is everywhere dense if and only if A is everywhere dense.

(63) For every open subspace X_0 of X and for all subsets A, C of X and for every subset B of X_0 such that $C =$ the carrier of X_0 and $A = B$ holds C is dense and B is everywhere dense if and only if A is everywhere dense.

(64) For all subsets C, A of X and for every subset B of X_0 such that $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is everywhere dense and B is everywhere dense if and only if A is everywhere dense.

(65) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if B is boundary, then A is boundary.

(66) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds if A is boundary, then B is boundary.

(67) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds A is boundary if and only if B is boundary.

(68) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if B is nowhere dense, then A is nowhere dense.

(69) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds if A is nowhere dense, then B is nowhere dense.

(70) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds A is nowhere dense if and only if B is nowhere dense.

4. Subsets in Topological Spaces with the same Topological Structures

In the sequel X_1, X_2 will be topological spaces. Next we state several propositions:

(71) If the carrier of X_1 = the carrier of X_2, then for every subset C_1 of X_1 and for every subset C_2 of X_2 holds $C_1 = C_2$ if and only if ${C_1}^{\rm c} = {C_2}^{\rm c}$.

(72) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is open if and only if C_2 is open, then the topological structure of X_1 = the topological structure of X_2.

(73) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is closed if and only if C_2 is closed, then the topological structure of X_1 = the topological structure of X_2.

(74) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\mathrm{Int}\, C_1 = \mathrm{Int}\, C_2$, then the topological structure of X_1 = the topological structure of X_2.

(75) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\overline{C_1} = \overline{C_2}$, then the topological structure of X_1 = the topological structure of X_2.

In the sequel D_1 is a subset of X_1 and D_2 is a subset of X_2. One can prove the following propositions:

(76) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is open, then D_2 is open.

(77) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\mathrm{Int}\, D_1 = \mathrm{Int}\, D_2$.

(78) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\mathrm{Int}\, D_1 \subseteq \mathrm{Int}\, D_2$.

(79) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is closed, then D_2 is closed.

(80) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} = \overline{D_2}$.

(81) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} \subseteq \overline{D_2}$.

(82) If $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is boundary, then D_2 is boundary.

(83) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is dense, then D_2 is dense.

(84) If $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is nowhere dense, then D_2 is nowhere dense.

(85) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is everywhere dense, then D_2 is everywhere dense.

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