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On a Duality Between Weakly Separated Subspaces of Topological Spaces

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Summary. Let X be a topological space and let X_1 and X_2 be subspaces of X with the carriers A_1 and A_2, respectively. Recall that X_1 and X_2 are *weakly separated* if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated (see [2] and also [1] for applications). Our purpose is to list a number of properties of such subspaces, supplementary to those given in [2]. Note that in the Mizar formalism the carrier of any topological space (hence the carrier of any its subspace) is always non–empty, therefore for convenience we list beforehand analogous properties of weakly separated subsets without any additional conditions.

To present the main results we first formulate a useful definition. We say that X_1 and X_2 *constitute a decomposition* of X if A_1 and A_2 are disjoint and the union of A_1 and A_2 covers the carrier of X (comp. [3]). We are ready now to present the following duality property between pairs of weakly separated subspaces : *If each pair of subspaces X_1, Y_1 and X_2, Y_2 of X constitutes a decomposition of X, then X_1 and X_2 are weakly separated iff Y_1 and Y_2 are weakly separated.* From this theorem we get immediately that under the same hypothesis, *X_1 and X_2 are separated iff X_1 misses X_2 and Y_1 and Y_2 are weakly separated.* Moreover, we show the following enlargement theorem : *If X_i and Y_i are subspaces of X such that Y_i is a subspace of X_i and $Y_1 \cup Y_2 = X_1 \cup X_2$ and if Y_1 and Y_2 are weakly separated, then X_1 and X_2 are weakly separated.* We show also the following dual extenuation theorem : *If X_i and Y_i are subspaces of X such that Y_i is a subspace of X_i and $Y_1 \cap Y_2 = X_1 \cap X_2$ and if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.* At the end we give a few properties of weakly separated subspaces in subspaces.

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The papers [6], [7], [4], [8], [5], and [2] provide the notation and terminology for this paper.

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1. Certain Set–Decompositions of a Topological Space

In the sequel X denotes a topological space. Next we state the proposition

(1) For all subsets A, B of X holds $A^{c} \setminus B^{c} = B \setminus A$.

Let X be a topological space, and let A_1, A_2 be subsets of X. We say that A_1 and A_2 constitute a decomposition if and only if:

(Def.1) $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 =$ the carrier of X.

In the sequel A, A_1, A_2, B_1, B_2 are subsets of X. We now state a number of propositions:

(2) A_1 and A_2 constitute a decomposition if and only if $A_1 \cap A_2 = \emptyset_X$ and $A_1 \cup A_2 = \Omega_X$.

(3) If A_1 and A_2 constitute a decomposition, then A_2 and A_1 constitute a decomposition.

(4) If A_1 and A_2 constitute a decomposition, then $A_1 = {A_2}^{c}$ and $A_2 = {A_1}^{c}$.

(5) If $A_1 = {A_2}^{c}$ or $A_2 = {A_1}^{c}$, then A_1 and A_2 constitute a decomposition.

(6) A and A^{c} constitute a decomposition and A^{c} and A constitute a decomposition.

(7) $\emptyset_X$ and Ω_X constitute a decomposition and Ω_X and $\emptyset_X$ constitute a decomposition.

(8) If A is non-empty, then A and A do not constitute a decomposition.

(9) If A_1 and A constitute a decomposition and A and A_2 constitute a decomposition, then $A_1 = A_2$.

(10) If A_1 and A_2 constitute a decomposition, then $\overline{A_1}$ and $\operatorname{Int} A_2$ constitute a decomposition and $\operatorname{Int} A_1$ and $\overline{A_2}$ constitute a decomposition.

(11) $\overline{A}$ and $\operatorname{Int}(A^{c})$ constitute a decomposition and $\overline{A^{c}}$ and $\operatorname{Int} A$ constitute a decomposition and $\operatorname{Int} A$ and $\overline{A^{c}}$ constitute a decomposition and $\operatorname{Int}(A^{c})$ and $\overline{A}$ constitute a decomposition.

(12) If A_1 and A_2 constitute a decomposition, then A_1 is open if and only if A_2 is closed.

(13) If A_1 and A_2 constitute a decomposition, then A_1 is closed if and only if A_2 is open.

(14) If A_1 and A_2 constitute a decomposition and B_1 and B_2 constitute a decomposition, then $A_1 \cap B_1$ and $A_2 \cup B_2$ constitute a decomposition.

(15) If A_1 and A_2 constitute a decomposition and B_1 and B_2 constitute a decomposition, then $A_1 \cup B_1$ and $A_2 \cap B_2$ constitute a decomposition.

2. Duality between Pairs of Weakly Separated Subsets

In the sequel X will denote a topological space and A_1, A_2 will denote subsets of X. Next we state a number of propositions:

(16) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds A_1 and A_2 are weakly separated if and only if C_1 and C_2 are weakly separated.

(17) A_1 and A_2 are weakly separated if and only if ${A_1}^{\mathrm{c}}$ and ${A_2}^{\mathrm{c}}$ are weakly separated.

(18) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if A_1 and A_2 are separated, then C_1 and C_2 are weakly separated.

(19) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if $A_1 \cap A_2 = \emptyset$ and C_1 and C_2 are weakly separated, then A_1 and A_2 are separated.

(20) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if $C_1 \cup C_2 =$ the carrier of X and C_1 and C_2 are weakly separated, then A_1 and A_2 are separated.

(21) If A_1 and A_2 constitute a decomposition, then A_1 and A_2 are weakly separated if and only if A_1 and A_2 are separated.

(22) A_1 and A_2 are weakly separated if and only if $(A_1 \cup A_2) \setminus A_1$ and $(A_1 \cup A_2) \setminus A_2$ are separated.

(23) For all subsets A_1, A_2, C_1, C_2 of X such that $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C_1 \cup C_2 = A_1 \cup A_2$ holds if C_1 and C_2 are weakly separated, then A_1 and A_2 are weakly separated.

(24) A_1 and A_2 are weakly separated if and only if $A_1 \setminus A_1 \cap A_2$ and $A_2 \setminus A_1 \cap A_2$ are separated.

(25) For all subsets A_1, A_2, C_1, C_2 of X such that $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C_1 \cap C_2 = A_1 \cap A_2$ holds if A_1 and A_2 are weakly separated, then C_1 and C_2 are weakly separated.

In the sequel X_0 will denote a subspace of X and B_1, B_2 will denote subsets of X_0. One can prove the following propositions:

(26) If $B_1 = A_1$ and $B_2 = A_2$, then A_1 and A_2 are separated if and only if B_1 and B_2 are separated.

(27) If $B_1 =$ (the carrier of X_0) $\cap A_1$ and $B_2 =$ (the carrier of X_0) $\cap A_2$, then if A_1 and A_2 are separated, then B_1 and B_2 are separated.

(28) If $B_1 = A_1$ and $B_2 = A_2$, then A_1 and A_2 are weakly separated if and only if B_1 and B_2 are weakly separated.

(29) If $B_1 =$ (the carrier of X_0) $\cap A_1$ and $B_2 =$ (the carrier of X_0) $\cap A_2$, then if A_1 and A_2 are weakly separated, then B_1 and B_2 are weakly separated.

3. Certain Subspace–Decompositions of a Topological Space

Let X be a topological space, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 constitute a decomposition if and only if:

(Def.2) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 constitute a decomposition.

In the sequel X_0, X_1, X_2, Y_1, Y_2 denote subspaces of X. The following propositions are true:

(30) X_1 and X_2 constitute a decomposition if and only if X_1 misses X_2 and the topological structure of $X = X_1 \cup X_2$.

(31) If X_1 and X_2 constitute a decomposition, then X_2 and X_1 constitute a decomposition.

(32) X_0 and X_0 do not constitute a decomposition.

(33) If X_1 and X_0 constitute a decomposition and X_0 and X_2 constitute a decomposition, then the topological structure of $X_1 =$ the topological structure of X_2.

(34) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds $Y_1 \cup Y_2 =$ the topological structure of X if and only if X_1 misses X_2.

(35) If X_1 and X_2 constitute a decomposition, then X_1 is open if and only if X_2 is closed.

(36) If X_1 and X_2 constitute a decomposition, then X_1 is closed if and only if X_2 is open.

(37) If X_1 meets Y_1 and X_1 and X_2 constitute a decomposition and Y_1 and Y_2 constitute a decomposition, then $X_1 \cap Y_1$ and $X_2 \cup Y_2$ constitute a decomposition.

(38) If X_2 meets Y_2 and X_1 and X_2 constitute a decomposition and Y_1 and Y_2 constitute a decomposition, then $X_1 \cup Y_1$ and $X_2 \cap Y_2$ constitute a decomposition.

4. Duality between Pairs of Weakly Separated Subspaces

In the sequel X is a topological space. We now state several propositions:

(39) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds X_1 and X_2 are weakly separated if and only if Y_1 and Y_2 are weakly separated.

(40) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if X_1 and X_2 are separated, then Y_1 and Y_2 are weakly separated.

(41) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if X_1

misses X_2 and Y_1 and Y_2 are weakly separated, then X_1 and X_2 are separated.

(42) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if $Y_1 \cup Y_2 =$ the topological structure of X and Y_1 and Y_2 are weakly separated, then X_1 and X_2 are separated.

(43) For all subspaces X_1, X_2 of X such that X_1 and X_2 constitute a decomposition holds X_1 and X_2 are weakly separated if and only if X_1 and X_2 are separated.

(44) For all subspaces X_1, X_2, Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and $Y_1 \cup Y_2 = X_1 \cup X_2$ holds if Y_1 and Y_2 are weakly separated, then X_1 and X_2 are weakly separated.

(45) For all subspaces X_1, X_2, Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and Y_1 meets Y_2 and $Y_1 \cap Y_2 = X_1 \cap X_2$ holds if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.

In the sequel X_0 will denote a subspace of X. Next we state four propositions:

(46) For all subspaces X_1, X_2 of X and for all subspaces Y_1, Y_2 of X_0 such that the carrier of $X_1 =$ the carrier of Y_1 and the carrier of $X_2 =$ the carrier of Y_2 holds X_1 and X_2 are separated if and only if Y_1 and Y_2 are separated.

(47) For all subspaces X_1, X_2 of X such that X_1 meets X_0 and X_2 meets X_0 and for all subspaces Y_1, Y_2 of X_0 such that $Y_1 = X_1 \cap X_0$ and $Y_2 = X_2 \cap X_0$ holds if X_1 and X_2 are separated, then Y_1 and Y_2 are separated.

(48) For all subspaces X_1, X_2 of X and for all subspaces Y_1, Y_2 of X_0 such that the carrier of $X_1 =$ the carrier of Y_1 and the carrier of $X_2 =$ the carrier of Y_2 holds X_1 and X_2 are weakly separated if and only if Y_1 and Y_2 are weakly separated.

(49) For all subspaces X_1, X_2 of X such that X_1 meets X_0 and X_2 meets X_0 and for all subspaces Y_1, Y_2 of X_0 such that $Y_1 = X_1 \cap X_0$ and $Y_2 = X_2 \cap X_0$ holds if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.

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