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Algebra of Vector Functions

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Summary. We develop the algebra of partial vector functions, with domains being algebra of vector functions.

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The terminology and notation used in this paper have been introduced in the following papers: [10], [5], [2], [3], [1], [12], [9], [4], [6], [11], [8], and [7]. For simplicity we adopt the following rules: X, Y will denote sets, C will denote a non-empty set, c will denote an element of C, V will denote a real normed space, f, f_1, f_2, f_3 will denote partial functions from C to the carrier of V, and r, p will denote real numbers. We now define several new functors. Let us consider C, V, f_1, f_2. The functor $f_1 + f_2$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.1) $\operatorname{dom}(f_1 + f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yields a partial function from C to the carrier of V and is defined as follows:

(Def.2) $\operatorname{dom}(f_1 - f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

Let us consider C, and let us consider V, and let f_1 be a partial function from C to $\mathbb{R}$, and let us consider f_2. The functor $f_1\, f_2$ yielding a partial function from C to the carrier of V is defined by:

(Def.3) $\operatorname{dom}(f_1\, f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom}(f_1\, f_2)$ holds $(f_1\, f_2)(c) = f_1(c) \cdot f_2(c)$.

Let us consider C, V, f, r. The functor $r\, f$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.4) $\operatorname{dom}(r\, f) = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}(r\, f)$ holds $(r\, f)(c) = r \cdot f(c)$.

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Let us consider C, V, f. The functor $\|f\|$ yields a partial function from C to $\mathbb{R}$ and is defined by:

(Def.5) $\operatorname{dom}\|f\| = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}\|f\|$ holds $\|f\|(c) = \|f(c)\|$.

The functor $-f$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.6) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}(-f)$ holds $(-f)(c) = -f(c)$.

Next we state a number of propositions:

(1) $f = f_1 + f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) + f_2(c)$.

(2) $f = f_1 - f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) - f_2(c)$.

(3) For every partial function f_1 from C to $\mathbb{R}$ holds $f = f_1\, f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) \cdot f_2(c)$.

(4) $f = r\, f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = r \cdot f_1(c)$.

(5) For every partial function f from C to $\mathbb{R}$ holds $f = \|f_1\|$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = \|f_1(c)\|$.

(6) $f = -f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = -f_1(c)$.

(7) For every partial function f_1 from C to $\mathbb{R}$ holds $\operatorname{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1} \{0_V\} = (\operatorname{dom} f_1 \setminus f_1^{-1} \{0\}) \cap (\operatorname{dom} f_2 \setminus f_2^{-1} \{0_V\})$.

(8) $\|f\|^{-1} \{0\} = f^{-1} \{0_V\}$ and $(-f)^{-1} \{0_V\} = f^{-1} \{0_V\}$.

(9) If $r \neq 0$, then $(r\, f)^{-1} \{0_V\} = f^{-1} \{0_V\}$.

(10) $f_1 + f_2 = f_2 + f_1$.

(11) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(12) For all partial functions f_1, f_2 from C to $\mathbb{R}$ and for every partial function f_3 from C to the carrier of V holds $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(13) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 + f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(14) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, (f_1 + f_2) = f_3\, f_1 + f_3\, f_2$.

(15) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(16) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(17) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 - f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

(18) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, f_1 - f_3\, f_2 = f_3\, (f_1 - f_2)$.

(19) $r\, (f_1 + f_2) = r\, f_1 + r\, f_2$.

(20) $(r \cdot p)\, f = r\,(p\, f)$.

(21) $r\,(f_1 - f_2) = r\, f_1 - r\, f_2$.

(22) $f_1 - f_2 = (-1)\,(f_2 - f_1)$.

(23) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(24) $1\, f = f$.

(25) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(26) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(27) For every partial function f_1 from C to $\mathbb{R}$ holds $\|f_1\, f_2\| = |f_1|\, \|f_2\|$.

(28) $\|r\, f\| = |r|\, \|f\|$.

(29) $-f = (-1)\, f$.

(30) $--f = f$.

(31) $f_1 - f_2 = f_1 + -f_2$.

We now state a number of propositions:

(32) $f_1 - -f_2 = f_1 + f_2$.

(33) $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2 \restriction X$ and $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2$ and $(f_1 + f_2) \restriction X = f_1 + f_2 \restriction X$.

(34) For every partial function f_1 from C to $\mathbb{R}$ holds $(f_1\, f_2) \restriction X = (f_1 \restriction X)\,(f_2 \restriction X)$ and $(f_1\, f_2) \restriction X = (f_1 \restriction X)\, f_2$ and $(f_1\, f_2) \restriction X = f_1\,(f_2 \restriction X)$.

(35) $(-f) \restriction X = -f \restriction X$ and $\|f\| \restriction X = \|f \restriction X\|$.

(36) $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2 \restriction X$ and $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2$ and $(f_1 - f_2) \restriction X = f_1 - f_2 \restriction X$.

(37) $(r\, f) \restriction X = r\,(f \restriction X)$.

(38) f_1 is total and f_2 is total if and only if $f_1 + f_2$ is total and also f_1 is total and f_2 is total if and only if $f_1 - f_2$ is total.

(39) For every partial function f_1 from C to $\mathbb{R}$ holds f_1 is total and f_2 is total if and only if $f_1\, f_2$ is total.

(40) f is total if and only if $r\, f$ is total.

(41) f is total if and only if $-f$ is total.

(42) f is total if and only if $\|f\|$ is total.

(43) If f_1 is total and f_2 is total, then $(f_1 + f_2)(c) = f_1(c) + f_2(c)$ and $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

(44) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is total and f_2 is total holds $(f_1\, f_2)(c) = f_1(c) \cdot f_2(c)$.

(45) If f is total, then $(r\, f)(c) = r \cdot f(c)$.

(46) If f is total, then $(-f)(c) = -f(c)$ and $\|f\|(c) = \|f(c)\|$.

Let us consider C, V, f, Y. We say that f is bounded on Y if and only if:

(Def.7) there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $\|f(c)\| \leq r$.

Next we state a number of propositions:

(47) f is bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $\|f(c)\| \leq r$.

(48) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(49) If $X \cap \operatorname{dom} f = \emptyset$, then f is bounded on X.

(50) If $0 = r$, then $r\, f$ is bounded on Y.

(51) If f is bounded on Y, then $r\, f$ is bounded on Y.

(52) If f is bounded on Y, then $\|f\|$ is bounded on Y and $-f$ is bounded on Y.

(53) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(54) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is bounded on X and f_2 is bounded on Y holds $f_1\, f_2$ is bounded on $X \cap Y$.

(55) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 - f_2$ is bounded on $X \cap Y$.

(56) If f is bounded on X and f is bounded on Y, then f is bounded on $X \cup Y$.

(57) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$.

(58) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is a constant on X and f_2 is a constant on Y holds $f_1\, f_2$ is a constant on $X \cap Y$.

(59) If f is a constant on Y, then $p\, f$ is a constant on Y.

(60) If f is a constant on Y, then $\|f\|$ is a constant on Y and $-f$ is a constant on Y.

(61) If f is a constant on Y, then f is bounded on Y.

(62) If f is a constant on Y, then for every r holds $r\, f$ is bounded on Y and $-f$ is bounded on Y and $\|f\|$ is bounded on Y.

(63) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(64) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$.

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