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Representation Theorem for Boolean Algebras

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The notation and terminology used in this paper are introduced in the following articles: [9], [7], [4], [5], [3], [10], [11], [8], [12], [1], [2], and [6].

In the sequel T is a topological space, X, Y are subsets of T, and x is arbitrary.

Let T be a topological space. The functor OpenClosedSet(T) yielding a non empty family of subsets of the carrier of T is defined as follows:

(Def.1) OpenClosedSet(T) $= \{x : x$ ranges over subsets of T, x is open $\land$ x is closed$\}$.

The following propositions are true:

(1) If $x \in$ OpenClosedSet(T), then there exists X such that $X = x$.

(2) If $X \in$ OpenClosedSet(T), then X is open.

(3) If $X \in$ OpenClosedSet(T), then X is closed.

(4) If X is open and closed, then $X \in$ OpenClosedSet(T).

Let X be a non empty set and let t be a non empty family of subsets of X. We see that the element of t is a subset of X.

In the sequel x, y, z will denote elements of OpenClosedSet(T).

Let us consider T and let C, D be elements of OpenClosedSet(T). Then $C \cup D$ is an element of OpenClosedSet(T).

Let us consider T and let C, D be elements of OpenClosedSet(T). Then $C \cap D$ is an element of OpenClosedSet(T).

Let us consider T. The functor join(T) yielding a binary operation on OpenClosedSet(T) is defined by:

(Def.2) For all elements A, B of OpenClosedSet(T) holds $(\text{join}(T))(A, B) = A \cup B$.

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Let us consider T. The functor meet(T) yields a binary operation on OpenClosedSet(T) and is defined by:

(Def.3) For all elements A, B of OpenClosedSet(T) holds $(\text{meet}(T))(A, B) = A \cap B$.

We now state several propositions:

(5) Let x, y be elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and let x', y' be elements of OpenClosedSet(T). If $x = x'$ and $y = y'$, then $x \sqcup y = x' \cup y'$.

(6) Let x, y be elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and let x', y' be elements of OpenClosedSet(T). If $x = x'$ and $y = y'$, then $x \sqcap y = x' \cap y'$.

(7) $\emptyset_T$ is an element of OpenClosedSet(T).

(8) Ω_T is an element of OpenClosedSet(T).

(9) For every element x of OpenClosedSet(T) holds x^c is an element of OpenClosedSet(T).

(10) $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ is a lattice.

Let T be a topological space. The functor OpenClosedSetLatt(T) yields a lattice and is defined by:

(Def.4) OpenClosedSetLatt(T) = $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$.

Next we state two propositions:

(11) For every topological space T and for all elements x, y of the carrier of OpenClosedSetLatt(T) holds $x \sqcup y = x \cup y$.

(12) For every topological space T and for all elements x, y of the carrier of OpenClosedSetLatt(T) holds $x \sqcap y = x \cap y$.

We follow a convention: a, b, c denote elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and x, y, z denote elements of OpenClosedSet(T).

The following propositions are true:

(13) The carrier of OpenClosedSetLatt(T) = OpenClosedSet(T).

(14) OpenClosedSetLatt(T) is Boolean.

(15) Ω_T is an element of the carrier of OpenClosedSetLatt(T).

(16) $\emptyset_T$ is an element of the carrier of OpenClosedSetLatt(T).

One can check that there exists a Boolean lattice which is non trivial.

For simplicity we adopt the following convention: L_1, L_2 denote lattices, a, p, q' denote elements of the carrier of B_1, U_1 denotes a filter of B_1, B denotes a subset of the carrier of B_1, and D denotes a non empty subset of the carrier of B_1.

Let us consider B_1. The functor ultraset(B_1) yields a non empty subset of $2^{\text{the carrier of } B_1}$ and is defined by:

(Def.5) ultraset(B_1) = $\{F : F \text{ is ultrafilter}\}$.

Next we state two propositions:

(18)[1] $x \in \mathrm{ultraset}(B_1)$ iff there exists U_1 such that $U_1 = x$ and U_1 is ultrafilter.

(19) For every a holds $\{F : F$ is ultrafilter $\wedge\ a \in F\} \subseteq \mathrm{ultraset}(B_1)$.

Let us consider B_1. The functor $\mathrm{UFilter}(B_1)$ yielding a function is defined as follows:

(Def.6) $\mathrm{dom\,UFilter}(B_1)$ = the carrier of B_1 and for every element a of the carrier of B_1 holds $(\mathrm{UFilter}(B_1))(a) = \{U_1 : U_1$ is ultrafilter $\wedge\ a \in U_1\}$.

Next we state several propositions:

(20) $x \in (\mathrm{UFilter}(B_1))(a)$ iff there exists F such that $F = x$ and F is ultrafilter and $a \in F$.

(21) $F \in (\mathrm{UFilter}(B_1))(a)$ iff F is ultrafilter and $a \in F$.

(22) For every F such that F is ultrafilter holds $a \sqcup b \in F$ iff $a \in F$ or $b \in F$.

(23) $(\mathrm{UFilter}(B_1))(a \sqcap b) = (\mathrm{UFilter}(B_1))(a) \cap (\mathrm{UFilter}(B_1))(b)$.

(24) $(\mathrm{UFilter}(B_1))(a \sqcup b) = (\mathrm{UFilter}(B_1))(a) \cup (\mathrm{UFilter}(B_1))(b)$.

Let us consider B_1. Then $\mathrm{UFilter}(B_1)$ is a function from the carrier of B_1 into $2^{\mathrm{ultraset}(B_1)}$.

Let us consider B_1. The functor $\mathrm{StoneR}(B_1)$ yielding a non empty set is defined as follows:

(Def.7) $\mathrm{StoneR}(B_1) = \mathrm{rng\,UFilter}(B_1)$.

The following propositions are true:

(25) $\mathrm{StoneR}(B_1) \subseteq 2^{\mathrm{ultraset}(B_1)}$.

(26) $x \in \mathrm{StoneR}(B_1)$ iff there exists a such that $(\mathrm{UFilter}(B_1))(a) = x$.

Let us consider B_1. The functor $\mathrm{StoneSpace}(B_1)$ yielding a strict topological space is defined by:

(Def.8) The carrier of $\mathrm{StoneSpace}(B_1) = \mathrm{ultraset}(B_1)$ and the topology of $\mathrm{StoneSpace}(B_1) = \{\bigcup A : A$ ranges over subsets of $2^{\mathrm{ultraset}(B_1)}$, $A \subseteq \mathrm{StoneR}(B_1)\}$.

One can prove the following two propositions:

(27) If F is ultrafilter and $F \notin (\mathrm{UFilter}(B_1))(a)$, then $a \notin F$.

(28) $\mathrm{ultraset}(B_1) \setminus (\mathrm{UFilter}(B_1))(a) = (\mathrm{UFilter}(B_1))(a^{\mathrm{c}})$.

Let us consider B_1. The functor $\mathrm{StoneBLattice}(B_1)$ yields a lattice and is defined as follows:

(Def.9) $\mathrm{StoneBLattice}(B_1) = \mathrm{OpenClosedSetLatt}(\mathrm{StoneSpace}(B_1))$.

One can prove the following four propositions:

(29) $\mathrm{UFilter}(B_1)$ is one-to-one.

(30) $\bigcup \mathrm{StoneR}(B_1) = \mathrm{ultraset}(B_1)$.

(31) For all sets A, B, X such that $X \subseteq \bigcup(A \cup B)$ and for arbitrary Y such that $Y \in B$ holds $Y \cap X = \emptyset$ holds $X \subseteq \bigcup A$.

(32) For every non empty set X holds there exists finite subset of X which is non empty.

[1]The proposition (17) has been removed.

Let D be a non empty set. Note that there exists a finite subset of D which is non empty.

The following propositions are true:

(33) For every lattice L and for all elements a, b, c, d of the carrier of L such that $a \sqsubseteq c$ and $b \sqsubseteq d$ holds $a \sqcap b \sqsubseteq c \sqcap d$.

(34) Let L be a non trivial Boolean lattice and let D be a non empty subset of the carrier of L. Suppose $\bot_L \in [D)$. Then there exists a non empty finite subset B of the carrier of L such that $B \subseteq D$ and $\sqcap^{\mathrm{f}}_B = \bot_L$.

(35) For every lower bound lattice L it is not true that there exists a filter F of L such that F is ultrafilter and $\bot_L \in F$.

(36) $(\mathrm{UFilter}(B_1))(\bot_{(B_1)}) = \emptyset$.

(37) $(\mathrm{UFilter}(B_1))(\top_{(B_1)}) = \mathrm{ultraset}(B_1)$.

(38) If $\mathrm{ultraset}(B_1) = \bigcup X$ and X is a subset of $\mathrm{StoneR}(B_1)$, then there exists a finite subset Y of X such that $\mathrm{ultraset}(B_1) = \bigcup Y$.

(39) If $x \in 2^X$ and $y \in 2^X$, then $x \cap y \in 2^X$.

(40) $\mathrm{StoneR}(B_1) = \mathrm{OpenClosedSet}(\mathrm{StoneSpace}(B_1))$.

Let us consider B_1. Then $\mathrm{UFilter}(B_1)$ is a homomorphism from B_1 to $\mathrm{StoneBLattice}(B_1)$.

Next we state four propositions:

(41) $\mathrm{rng}\,\mathrm{UFilter}(B_1) =$ the carrier of $\mathrm{StoneBLattice}(B_1)$.

(42) $\mathrm{UFilter}(B_1)$ is isomorphism.

(43) B_1 and $\mathrm{StoneBLattice}(B_1)$ are isomorphic.

(44) For every non trivial Boolean lattice B_1 there exists a topological space T such that B_1 and $\mathrm{OpenClosedSetLatt}(T)$ are isomorphic.

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