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Many-sorted Sets

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Summary. The article deals with parameterized families of sets. When treated in a similar way as sets (due to systematic overloading notation used for sets) they are called many sorted sets. For instance, if x and X are two many-sorted sets (with the same set of indices I) then relation $x \in X$ is defined as $\forall_{i \in I} x_i \in X_i$.

I was prompted by a remark in a paper by Tarlecki and Wirsing: "Throughout the paper we deal with many-sorted sets, functions, relations etc. ... We feel free to use any standard set-theoretic notation without explicit use of indices" [3, p.97]. The aim of this work was to check the feasibility of such approach in Mizar. It works.

Let us observe some peculiarities:

- empty set (i.e. the many sorted set with empty set of indices) belongs to itself (theorem 133),
- we get two different inclusions $X \subseteq Y$ iff $\forall_{i \in I} X_i \subseteq Y_i$ and $X \sqsubseteq Y$ iff $\forall_x x \in X \Rightarrow x \in Y$ equivalent only for sets that yield non empty values.

Therefore the care is advised.

MML Identifier: PBOOLE.

The articles [5], [1], [4], and [2] provide the terminology and notation for this paper.

1. Preliminaries

In the sequel i, e will be arbitrary.

A function is empty yielding if:

(Def.1) For every i such that $i \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(i)$ is empty.

A function is non empty set yielding if:

(Def.2) For every i such that $i \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(i)$ is non empty.

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Next we state two propositions:

(1) For every function f such that f is non empty yielding holds $\operatorname{rng} f$ has non empty elements.

(2) For every function f holds f is empty yielding iff $f = \emptyset$ or $\operatorname{rng} f = \{\emptyset\}$.

In the sequel I denotes a set.

Let us consider I. A function is said to be a many sorted set of I if:

(Def.3) $\operatorname{dom} \mathrm{it} = I$.

In the sequel x, y, z, X, Y, Z, V are many sorted sets of I.

The scheme *Kuratowski Function* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a many sorted set f of $\mathcal{A}$ such that for every e such that $e \in \mathcal{A}$ holds $f(e) \in \mathcal{F}(e)$

provided the following requirement is met:

- For every e such that $e \in \mathcal{A}$ holds $\mathcal{F}(e) \neq \emptyset$.

Let us consider I, X, Y. The predicate $X \in Y$ is defined by:

(Def.4) For every i such that $i \in I$ holds $X(i) \in Y(i)$.

The predicate $X \subseteq Y$ is defined by:

(Def.5) For every i such that $i \in I$ holds $X(i) \subseteq Y(i)$.

The scheme *PSeparation* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a many sorted set X of $\mathcal{A}$ such that for every set i holds if $i \in \mathcal{A}$, then for every e holds $e \in X(i)$ iff $e \in \mathcal{B}(i)$ and $\mathcal{P}[i, e]$

for all values of the parameters.

One can prove the following proposition

(3) If for every i such that $i \in I$ holds $X(i) = Y(i)$, then $X = Y$.

Let us consider I. The functor $\emptyset_I$ yields a many sorted set of I and is defined by:

(Def.6) $\emptyset_I = I \longmapsto \emptyset$.

Let us consider X, Y. The functor $X \cup Y$ yielding a many sorted set of I is defined by:

(Def.7) For every i such that $i \in I$ holds $(X \cup Y)(i) = X(i) \cup Y(i)$.

The functor $X \cap Y$ yielding a many sorted set of I is defined by:

(Def.8) For every i such that $i \in I$ holds $(X \cap Y)(i) = X(i) \cap Y(i)$.

The functor $X \setminus Y$ yields a many sorted set of I and is defined as follows:

(Def.9) For every i such that $i \in I$ holds $(X \setminus Y)(i) = X(i) \setminus Y(i)$.

We say that X overlaps Y if and only if:

(Def.10) For every i such that $i \in I$ holds $X(i)$ meets $Y(i)$.

We say that X misses Y if and only if:

(Def.11) For every i such that $i \in I$ holds $X(i)$ misses $Y(i)$.

Let us consider I, X, Y. The functor $X \dot{-} Y$ yielding a many sorted set of I is defined as follows:

(Def.12) $X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X)$.

Next we state several propositions:

(4) For every i such that $i \in I$ holds $(X \dot{-} Y)(i) = X(i) \dot{-} Y(i)$.

(5) For every i such that $i \in I$ holds $\emptyset_I(i) = \emptyset$.

(6) If for every i such that $i \in I$ holds $X(i) = \emptyset$, then $X = \emptyset_I$.

(7) If $x \in X$ or $x \in Y$, then $x \in X \cup Y$.

(8) $x \in X \cap Y$ iff $x \in X$ and $x \in Y$.

(9) If $x \in X$ and $X \subseteq Y$, then $x \in Y$.

(10) If $x \in X$ and $x \in Y$, then X overlaps Y.

(11) If X overlaps Y, then there exists x such that $x \in X$ and $x \in Y$.

(12) If $x \in X \setminus Y$, then $x \in X$.

2. Lattice Properties of Many Sorted Sets

One can prove the following proposition

(13) $X \subseteq X$.

Let us consider I, X, Y. Let us observe that $X = Y$ if and only if:

(Def.13) $X \subseteq Y$ and $Y \subseteq X$.

Next we state a number of propositions:

(14) If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

(15) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(16) $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.

(17) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

(18) If $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

(19) If $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$.

(20) If $X \subseteq Y$, then $X \cup Z \subseteq Y \cup Z$ and $Z \cup X \subseteq Z \cup Y$.

(21) If $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$ and $Z \cap X \subseteq Z \cap Y$.

(22) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cup Z \subseteq Y \cup V$.

(23) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cap Z \subseteq Y \cap V$.

(24) If $X \subseteq Y$, then $X \cup Y = Y$ and $Y \cup X = Y$.

(25) If $X \subseteq Y$, then $X \cap Y = X$ and $Y \cap X = X$.

(26) $X \cap Y \subseteq X \cup Z$.

(27) If $X \subseteq Z$, then $X \cup Y \cap Z = (X \cup Y) \cap Z$.

(28) $X = Y \cup Z$ iff $Y \subseteq X$ and $Z \subseteq X$ and for every V such that $Y \subseteq V$ and $Z \subseteq V$ holds $X \subseteq V$.

(29) $X = Y \cap Z$ iff $X \subseteq Y$ and $X \subseteq Z$ and for every V such that $V \subseteq Y$ and $V \subseteq Z$ holds $V \subseteq X$.

(30) $X \cup X = X$.

(31) $X \cap X = X$.

(32) $X \cup Y = Y \cup X$.

(33) $X \cap Y = Y \cap X$.

(34) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

(35) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

(36) $X \cap (X \cup Y) = X$ and $(X \cup Y) \cap X = X$ and $X \cap (Y \cup X) = X$ and $(Y \cup X) \cap X = X$.

(37) $X \cup X \cap Y = X$ and $X \cap Y \cup X = X$ and $X \cup Y \cap X = X$ and $Y \cap X \cup X = X$.

(38) $X \cap (Y \cup Z) = X \cap Y \cup X \cap Z$ and $(Y \cup Z) \cap X = Y \cap X \cup Z \cap X$.

(39) $X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z)$ and $Y \cap Z \cup X = (Y \cup X) \cap (Z \cup X)$.

(40) If $X \cap Y \cup X \cap Z = X$, then $X \subseteq Y \cup Z$.

(41) If $(X \cup Y) \cap (X \cup Z) = X$, then $Y \cap Z \subseteq X$.

(42) $X \cap Y \cup Y \cap Z \cup Z \cap X = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X)$.

(43) If $X \cup Y \subseteq Z$, then $X \subseteq Z$ and $Y \subseteq Z$.

(44) If $X \subseteq Y \cap Z$, then $X \subseteq Y$ and $X \subseteq Z$.

(45) $(X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z)$ and $X \cup (Y \cup Z) = (X \cup Y) \cup (X \cup Z)$.

(46) $(X \cap Y) \cap Z = X \cap Z \cap (Y \cap Z)$ and $X \cap (Y \cap Z) = (X \cap Y) \cap (X \cap Z)$.

(47) $X \cup (X \cup Y) = X \cup Y$ and $X \cup Y \cup Y = X \cup Y$.

(48) $X \cap (X \cap Y) = X \cap Y$ and $X \cap Y \cap Y = X \cap Y$.

3. The Empty Many Sorted Set

Next we state several propositions:

(49) $\emptyset_I \subseteq X$.

(50) If $X \subseteq \emptyset_I$, then $X = \emptyset_I$.

(51) If $X \subseteq Y$ and $X \subseteq Z$ and $Y \cap Z = \emptyset_I$, then $X = \emptyset_I$.

(52) If $X \subseteq Y$ and $Y \cap Z = \emptyset_I$, then $X \cap Z = \emptyset_I$.

(53) $X \cup \emptyset_I = X$ and $\emptyset_I \cup X = X$.

(54) If $X \cup Y = \emptyset_I$, then $X = \emptyset_I$ and $Y = \emptyset_I$.

(55) $X \cap \emptyset_I = \emptyset_I$ and $\emptyset_I \cap X = \emptyset_I$.

(56) If $X \subseteq Y \cup Z$ and $X \cap Z = \emptyset_I$, then $X \subseteq Y$.

(57) If $Y \subseteq X$ and $X \cap Y = \emptyset_I$, then $Y = \emptyset_I$.

4. The Difference and the Symmetric Difference

We now state a number of propositions:

(58) $X \setminus Y = \emptyset_I$ iff $X \subseteq Y$.

(59) If $X \subseteq Y$, then $X \setminus Z \subseteq Y \setminus Z$.

(60) If $X \subseteq Y$, then $Z \setminus Y \subseteq Z \setminus X$.

(61) If $X \subseteq Y$ and $Z \subseteq V$, then $X \setminus V \subseteq Y \setminus Z$.

(62) $X \setminus Y \subseteq X$.

(63) If $X \subseteq Y \setminus X$, then $X = \emptyset_I$.

(64) $X \setminus X = \emptyset_I$.

(65) $X \setminus \emptyset_I = X$.

(66) $\emptyset_I \setminus X = \emptyset_I$.

(67) $X \setminus (X \cup Y) = \emptyset_I$ and $X \setminus (Y \cup X) = \emptyset_I$.

(68) $X \cap (Y \setminus Z) = X \cap Y \setminus Z$.

(69) $(X \setminus Y) \cap Y = \emptyset_I$ and $Y \cap (X \setminus Y) = \emptyset_I$.

(70) $X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z$.

(71) $(X \setminus Y) \cup X \cap Y = X$ and $X \cap Y \cup (X \setminus Y) = X$.

(72) If $X \subseteq Y$, then $Y = X \cup (Y \setminus X)$ and $Y = (Y \setminus X) \cup X$.

(73) $X \cup (Y \setminus X) = X \cup Y$ and $(Y \setminus X) \cup X = Y \cup X$.

(74) $X \setminus (X \setminus Y) = X \cap Y$.

(75) $X \setminus Y \cap Z = (X \setminus Y) \cup (X \setminus Z)$.

(76) $X \setminus X \cap Y = X \setminus Y$ and $X \setminus Y \cap X = X \setminus Y$.

(77) $X \cap Y = \emptyset_I$ iff $X \setminus Y = X$.

(78) $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$.

(79) $X \setminus Y \setminus Z = X \setminus (Y \cup Z)$.

(80) $X \cap Y \setminus Z = (X \setminus Z) \cap (Y \setminus Z)$.

(81) $(X \cup Y) \setminus Y = X \setminus Y$.

(82) If $X \subseteq Y \cup Z$, then $X \setminus Y \subseteq Z$ and $X \setminus Z \subseteq Y$.

(83) $(X \cup Y) \setminus X \cap Y = (X \setminus Y) \cup (Y \setminus X)$.

(84) $X \setminus Y \setminus Y = X \setminus Y$.

(85) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.

(86) If $X \setminus Y = Y \setminus X$, then $X = Y$.

(87) $X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z$ and $(Y \setminus Z) \cap X = Y \cap X \setminus Z \cap X$.

(88) If $X \setminus Y \subseteq Z$, then $X \subseteq Y \cup Z$.

(89) $X \setminus Y \subseteq X \dot{-} Y$.

(90) $X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X)$.

(91) $X \dot{-} \emptyset_I = X$ and $\emptyset_I \dot{-} X = X$.

(92) $X \dot{-} X = \emptyset_I$.

(93) $X \dot{-} Y = Y \dot{-} X$.
(94) $X \cup Y = (X \dot{-} Y) \cup X \cap Y$.
(95) $X \dot{-} Y = (X \cup Y) \setminus X \cap Y$.
(96) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z))$.
(97) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \cup Z)) \cup X \cap Y \cap Z$.
(98) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.
(99) If $X \setminus Y \subseteq Z$ and $Y \setminus X \subseteq Z$, then $X \dot{-} Y \subseteq Z$.
(100) $X \cup Y = X \dot{-} (Y \setminus X)$.
(101) $X \cap Y = X \dot{-} (X \setminus Y)$.
(102) $X \setminus Y = X \dot{-} X \cap Y$.
(103) $Y \setminus X = X \dot{-} (X \cup Y)$.
(104) $X \cup Y = X \dot{-} Y \dot{-} X \cap Y$.
(105) $X \cap Y = X \dot{-} Y \dot{-} (X \cup Y)$.

5. Meeting and Overlapping

The following propositions are true:
(106) If X overlaps Y or X overlaps Z, then X overlaps $Y \cup Z$.
(107) If X overlaps Y, then Y overlaps X.
(108) If X overlaps Y and $Y \subseteq Z$, then X overlaps Z.
(109) If X overlaps Y and $X \subseteq Z$, then Z overlaps Y.
(110) If $X \subseteq Y$ and $Z \subseteq V$ and X overlaps Z, then Y overlaps V.
(111) If X overlaps $Y \cap Z$, then X overlaps Y and X overlaps Z.
(112) If X overlaps Z and $X \subseteq V$, then X overlaps $Z \cap V$.
(113) If X overlaps $Y \setminus Z$, then X overlaps Y.
(114) If Y does not overlap Z, then $X \cap Y$ does not overlap $X \cap Z$ and $Y \cap X$ does not overlap $Z \cap X$.
(115) If X overlaps $Y \setminus Z$, then Y overlaps $X \setminus Z$.
(116) If X meets Y and $Y \subseteq Z$, then X meets Z.
(117) If X meets Y, then Y meets X.
(118) Y misses $X \setminus Y$.
(119) $X \cap Y$ misses $X \setminus Y$.
(120) $X \cap Y$ misses $X \dot{-} Y$.
(121) If X misses Y, then $X \cap Y = \emptyset_I$.
(122) If $X \neq \emptyset_I$, then X meets X.
(123) If $X \subseteq Y$ and $X \subseteq Z$ and Y misses Z, then $X = \emptyset_I$.
(124) If $Z \cup V = X \cup Y$ and X misses Z and Y misses V, then $X = V$ and $Y = Z$.

(125) If $Z \cup V = X \cup Y$ and Y misses Z and X misses V, then $X = Z$ and $Y = V$.

(126) If X misses Y, then $X \setminus Y = X$ and $Y \setminus X = Y$.

(127) If X misses Y, then $(X \cup Y) \setminus Y = X$ and $(X \cup Y) \setminus X = Y$.

(128) If $X \setminus Y = X$, then X misses Y and Y misses X.

(129) $X \setminus Y$ misses $Y \setminus X$.

6. The Second Inclusion

Let us consider I, X, Y. The predicate $X \sqsubseteq Y$ is defined as follows:

(Def.14) For every x such that $x \in X$ holds $x \in Y$.

The following three propositions are true:

(130) If $X \subseteq Y$, then $X \sqsubseteq Y$.

(131) $X \sqsubseteq X$.

(132) If $X \sqsubseteq Y$ and $Y \sqsubseteq Z$, then $X \sqsubseteq Z$.

7. Non Empty and Non-empty Many Sorted Sets

The following propositions are true:

(133) $\emptyset_{\emptyset} \in \emptyset_{\emptyset}$.

(134) For every many sorted set X of $\emptyset$ holds $X = \emptyset$.

We follow a convention: I will be a non empty set and x, X, Y, Z will be many sorted sets of I.

The following propositions are true:

(135) If X overlaps Y, then X meets Y.

(136) It is not true that there exists x such that $x \in \emptyset_I$.

(137) If $x \in X$ and $x \in Y$, then $X \cap Y \neq \emptyset_I$.

(138) X does not overlap $\emptyset_I$ and $\emptyset_I$ does not overlap X.

(139) If $X \cap Y = \emptyset_I$, then X does not overlap Y.

(140) If X overlaps X, then $X \neq \emptyset_I$.

Let I be a set. A many sorted set of I is empty yielding if:

(Def.15) For every i such that $i \in I$ holds it(i) is empty.

A many sorted set of I is non empty set yielding if:

(Def.16) For every i such that $i \in I$ holds it(i) is non empty.

Let I be a non empty set. Observe that every many sorted set of I which is non-empty is also non empty and every many sorted set of I which is empty is also non non-empty.

One can prove the following propositions:

(141) X is empty iff $X = \emptyset_I$.

(142) If Y is empty and $X \subseteq Y$, then X is empty.

(143) If X is non-empty and $X \subseteq Y$, then Y is non-empty.

(144) If X is non-empty and $X \sqsubseteq Y$, then $X \subseteq Y$.

(145) If X is non-empty and $X \sqsubseteq Y$, then Y is non-empty.

In the sequel X denotes a non-empty many sorted set of I.

The following propositions are true:

(146) There exists x such that $x \in X$.

(147) If for every x holds $x \in X$ iff $x \in Y$, then $X = Y$.

(148) If for every x holds $x \in X$ iff $x \in Y$ and $x \in Z$, then $X = Y \cap Z$.

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